47. A continuous function ϕ on [a,b] is called *polygonal* (or *piecewise linear*) if there is a subdivision $a = x_0 < x_1 < \ldots < x_n$ such that ϕ is linear on each interval $[x_{n-1},x_n]$. Let f be an arbitrary continuous function on [a,b] and ε a positive number. Show that there is a polygonal function ϕ on [a,b] with $|f(x) - \phi(x)| < \varepsilon$ for all $x \in [a,b]$.

Proof.

Suppose f is continuous on [a,b] and $\varepsilon > 0$. Proposition 20 implies that f is uniformly continuous on [a,b]. Thus there is a $\delta > 0$ such that if x and y are in [a,b] with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2$. From the Axiom of Archimedes, there is a positive integer n such that $(b-a)/n < \delta$. Define a regular subdivision of [a,b] by setting $x_j = a + j\cdot(b-a)/n$ for j an integer with $0 \le j \le n$. Evidently $x_j - x_{j-1} = (b-a)/n < \delta$ for each j with $1 \le j \le n$.

Now we shall build ϕ. If $x \in [x_{j-1},x_j]$, set

$$\phi(x) = (x - x_{j-1})\cdot[(f(x_j) - f(x_{j-1}))/(x_j - x_{j-1})] + f(x_{j-1}).$$

Observe that there is no problem at the endpoints of the sub-intervals where we have overlap. Clearly ϕ is a polygonal function defined on [a,b].

Finally, we verify that ϕ satisfies the desired inequality. Let $x \in [a,b]$. Then there is some integer j with $1 \le j \le n$ such that $x \in [x_{j-1},x_j]$. Using the triangle inequality, we have

$$\begin{aligned} |f(x) - \phi(x)| &\le |f(x) - f(x_{j-1})| + |f(x_{j-1}) - \phi(x)| \\ &< (\varepsilon/2) + |f(x_j) - f(x_{j-1})| \\ &< (\varepsilon/2) + (\varepsilon/2) \\ &< \varepsilon \end{aligned}$$

since

$$|x - x_{j-1}| \le |x_j - x_{j-1}| = (b-a)/n < \delta$$

and

$$(x - x_{j-1})/(x_j - x_{j-1}) \le 1.$$