52. Let f be a lower semicontinuous function defined for all real numbers. What can you say about the sets $\{x : f(x) > \alpha\}$, $\{x : f(x) \geq \alpha\}$, $\{x : f(x) < \alpha\}$, $\{x : f(x) \leq \alpha\}$, and $\{x : f(x) = \alpha\}$.

Solution. From Problem 2-50, Part (c), if f is lower semicontinuous on all of the real line, the set $\{x : f(x) > \alpha\}$ must be open for each real number α. Observe that this is, in fact, $f^{-1}((\alpha,\infty))$. Since

$$\begin{aligned}\{x : f(x) \leq \alpha\} &= f^{-1}((-\infty,\alpha]) \\ &= f^{-1}(\mathbb{R} \sim (\alpha,\infty)) \\ &= f^{-1}(\mathbb{R}) \sim f^{-1}((\alpha,\infty)) \\ &= \mathbb{R} \sim \{x : f(x) > \alpha\},\end{aligned}$$

and $\{x : f(x) > \alpha\}$ is open, $\{x : f(x) \leq \alpha\}$ must be closed. Evidently,

$$\begin{aligned}\{x : f(x) \geq \alpha\} &= f^{-1}([\alpha,\infty)) \\ &= f^{-1}(\cap(\alpha - (1/n),\infty)) \\ &= \cap f^{-1}((\alpha - (1/n),\infty)),\end{aligned}$$

where the intersections are over the positive integers. Since f is lower semicontinuous,

$$f^{-1}((\alpha - (1/n),\infty)) = \{x : f(x) > \alpha -(1/n)\}$$

must be open for each positive integer n. Thus $\{x : f(x) \geq \alpha\}$ must be a G_δ. Similarly,

$$\begin{aligned}\{x : f(x) < \alpha\} &= f^{-1}((-\infty,\alpha)) \\ &= f^{-1}(\cup(-\infty,\alpha - (1/n)]) \\ &= \cup f^{-1}((-\infty,\alpha - (1/n)]),\end{aligned}$$

where the unions are over the positive integers. Since f is lower semicontinuous,

$$f^{-1}((-\infty,\alpha - (1/n)]) = \{x : f(x) \leq \alpha - (1/n)\}$$

must be closed for each positive integer n. Thus $\{x : f(x) < \alpha\}$ must be an F_σ. Finally we are left to deal with $\{x : f(x) = \alpha\}$. Evidently, $\{x : f(x) = \alpha\} = \{x : f(x) \leq \alpha\} \cap \{x : f(x) \geq \alpha\}$. The first set is closed and the second is a G_δ. Since each open set is also an F_σ, we may view $\{x : f(x) = \alpha\}$ as an $F_{\sigma\delta}$. Is it really something simpler --- like merely a G_δ?? $\{x : f(x) \leq \alpha\}$ is closed, and hence a G_δ. Thus $\{x : f(x) = \alpha\}$ is the intersection of two G_δ sets, and thus a G_δ.//