3-16. Show that, if A is any set with $m^*(A) > 0$, then there is a nonmeasurable set $E \cup A$.

Proof. Suppose A is any subset of the real line with $m^*(A) > 0$. Note that $A = A \cap \cup\{[n,n+1): n \in \mathbb{Z}\}$. Let $F_n = A \cap [n,n+1)$ for each integer, n. Let $<E_n>$ be a re-indexing of the family of sets $\{F_n\}$ using the natural numbers. Thus, from Problem 3-12, it follows that $m^*(A) = \sum m^*(E_n)$. Thus, there is an integer, j, such that $m^*(F_j) > 0$. Let $C = F_j - j$. Observe that $C \subset [0,1)$ and that, from the translation invariance of outer measure, $m^*(C) = m^*(F_j) > 0$.

We shall now show that C contains an unmeasurable set, which we will be able to translate back into A. Recall our friendly nonmeasurable subset P of [0,1) together with all its nice rational, pairwise disjoint translates, $<P_i>$, where $P_i = P + r_i$, where r_i is the ith rational number in [0,1), with $r_0 = 0$. Let $C_i = C \cap P_i$ for each i. Then $C = \cup C_i$ and the sequence of sets $<C_i>$ is pairwise disjoint. Evidently each C_i is a translate of a subset of P. If each C_i is measurable, then from Problem 3-15, the corresponding translate would have to be of measure zero. Thus, each C_i would have to be of measure zero from translation invariance. That, however, is impossible, for we have that $0 < m^*(C) \le \sum m^*(C_i)$. Thus, there must be an index i such that corresponding C_i is unmeasurable.

Finally, observe that if C_i is not measurable, then the same is true for $C_i + j$, which is a subset of F_j, which is a subset of the original varmint, A.//

3-18. Show that (v) does not imply (iv) in Proposition 3-18 by constructing a function f such that $\{x : f(x) > 0\} = E$, a given nonmeasurable set, and such that f assumes each value at most once.

Construction?? Let P be the nonmeasurable subset of [0,1) defined in Section 4 of Chapter 3. Set $E = P \sim \{0\}$ if $0 \in P$. Otherwise, simply let $E = P$. In either case E will not be a Lebesgue measurable set. Define $f:[0,1) \to \mathbb{R}$ by $f(x) = x$ if $x \in E$, and $f(x) = -2 + x$ if $x \in [0,1) \sim E$. Then f is one-to-one. As a consequence, $\{x: f(x) = \alpha\}$ is either a singleton or empty for each real number α, and thus, measurable. On the other hand, $\{x : f(x) > 0\}$ is not measurable. Thus (v) being true for each real number α does not imply that (iv) is true for all real α. [Keep in mind that (i)-(iv) are equivalent.]