3-2. Let $<E_n>$ be any sequence of sets in the collection of measurable sets. Then $m(\cup E_n) \le \sum m(E_n)$. [Hint: Use Proposition 1.2.] This property of a measure is called *countable subadditivity*.

Proof. From Proposition 1.2 and its proof, there is a sequence $<F_n>$ of pairwise disjoint measurable sets with $\cup E_n = \cup F_n$, and for each n, $F_n \subset E_n$. Thus, from the countable additivity of the measure and its monotonicity, Problem 3-1, we have
$m(\cup E_n) = m(\cup F_n) = \sum m(F_n) \le \sum m(E_n)$.//

3-5. Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a *finite* collection of open intervals covering A. Then $\sum l(I_n) \ge 1$.

Proof. Since A is a dense subset of [0,1], and $A \subset \cup I_n$, it follows from Proposition 2-10 that

$$[0,1] = \overline{A} \subset \overline{\cup I_n} = \cup \overline{I_n}.$$

Thus

$$m^*[0,1] = m^*(\overline{A}) \le m^*(\overline{\cup I_n}) \le \sum m^*(\overline{I_n}).$$

This, however, does the job since the outer measure of an interval is its length, from Proposition 3-1, so that the outer measure of the closure of each of interval I_n is its length, which is the same as the length of I_n.//

3-6. Prove Proposition 5.

5. Proposition: Given any set A and any $\varepsilon > 0$, there is an open set O such that $A \subset O$ and $m^*(O) \le m^*(A) + \varepsilon$. There is a $G \in G_\delta$ such that $A \subset G$ and $m^*(A) = m^*(G)$.

Proof. Let A be any subset of the real numbers. If $\{I_n\}$ is any countable collection of open intervals that cover A, then from Propositions 3-1 and 3-2, $m^*(O) \le \sum m^*(I_n) = \sum l(I_n)$, where $O = \cup I_n$ is clearly open.

Let $\varepsilon > 0$. If $m^*(A) = \infty$, then we don't need to do anything, for any countable cover of A by open intervals will do to provide an open set with the inequality $m^*(O) \le m^*(A) + \varepsilon$ satisfied, and $m^*(O) = m^*(A) = \infty$. [Open sets are also G_δ sets.] Thus, we need only consider the case where $m^*(A) < \infty$. Then, since $m^*(A) + \varepsilon$ is finite, from the definition of outer measure, there is a countable cover of A by open intervals, $\{I_n\}$, with $\sum l(I_n) < m^*(A) + \varepsilon$. If we set $O = \cup I_n$, then O is open, $A \subset O$, and from our observation in the first paragraph, we have that $m^*(O) \le m^*(A) + \varepsilon$. To produce, an appropriate G_δ set, for each n in $\mathbb{N}$, there is an open set O_n such that $A \subset O_n$ and $m^*(A) \le m^*(O_n) \le m^*(A) + (1/n)$. $G = \cap O_n$ plainly will do as the desired set.//