3-24. Let f be measurable and B a Borel set. Then $f^{-1}[B]$ is a measurable set. [Hint: The class of sets for which $f^{-1}[E]$ is measurable is a σ-algebra.]

Proof. Let $\mathscr{E} = \{ E \subset \mathbb{R} : f^{-1}[E]$ is Lebesgue measurable$\}$. Since inverse images are well behaved with respect to all the usual set operations and the family of Lebesgue measurable sets is a σ-algebra, it is easy to see that $\mathscr{E}$ is a σ-algebra. Here are the details of that. Suppose $E \in \mathscr{E}$. Then $f^{-1}[E]$ is Lebesgue measurable. Consequently, $\sim f^{-1}[E] = f^{-1}[\sim E]$ is Lebesgue measurable. Hence, $\sim E \in \mathscr{E}$. Similarly, if $\{E_j\}$ is a countable family of elements of E, then $\{f^{-1}[E_j]\}$ is a countable family of Lebesgue measurable sets. Consequently, $\cup f^{-1}[E_j] = f^{-1}[\cup E_j]$ is Lebesgue measurable. Hence $\cup E_j \in \mathscr{E}$. Since, $f^{-1}[\emptyset] = \emptyset$, we now know that $\mathscr{E}$ is a σ-algebra. Since the definition of measurability implies that $\mathscr{E}$ contains each open ray, (a,∞), $\mathscr{E}$ must contain the open intervals, and hence, all open subsets of the real line. Thus, $\mathscr{E}$ must contain the σ-algebra generated by the open sets, the Borel sets.//

3-25. Show that if f is a measurable real-valued function and g is a continuous function defined on $(-\infty,\infty)$, then $g \circ f$ is measurable.

Proof. If a is a real number, then

$$\begin{aligned}\{x: g\circ f(x) > a\} &= (g\circ f)^{-1}[(a,\infty)] \\ &= f^{-1}[g^{-1}[(a,\infty)]].\end{aligned}$$

If g is continuous, then $g^{-1}[(a,\infty)]$ is an open subset of $(-\infty,\infty)$. Open subsets are Borel sets. Now just apply Problem 3-24 with $B = g^{-1}[(a,\infty)]$.//