3-29. Give an example to show that we must require $m(E) < \infty$ in Proposition 23.

23. Proposition: Let E be a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable functions defined on E. Let f be a real-valued function such that for each x in E we have $f_n(x) \to f(x)$. Then given $\varepsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer N such that for all $x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

Example. For each n in $\mathbb{N}$ define $f_n:\mathbb{R} \to \mathbb{R}$ to be the characteristic function of the interval [n,n+1). Then each function in the sequence plainly is measurable, and the sequence $\langle f_n \rangle$ converges point-wise to the zero function, a measurable function. Let $\varepsilon_0 = 1/2$ and let $\delta_0 = 1/2$. If A is any measurable subset of the real line with $m(A) < \delta_0$ and N is any positive integer, since $m(A \cap [N,N+1)) < \delta_0$, there is a real number x_0 in $[N,N+1) \sim A$ with $|f_N(x_0) - 0| \geq \varepsilon_0$. Thus there is a positive integer $n \geq N$ and number $x_0 \notin A$ such that the inequality in the conclusion of Proposition 23 is false.

Note: Take a careful look at the proof of Proposition 3-23 given by Royden. The intuition is this: Since $\mathbb{R}$ has infinite measure, we can build a sequence that converges and yet have the measure of the "bad sets", the E_N's, not eventually vanish. We could, in fact, arrange it so that the bad sets get arbitrarily large in measure and the badness arbitrarily bad, while still converging point-wise to zero. Replace the sequence of functions, $\langle f_n \rangle$, above with the sequence of functions $\langle g_n \rangle$, where each function, g_n, is n! times the characteristic function of the interval [n, n + (n!)). Then given any $\varepsilon > 0$, any $\delta > 0$, any measurable subset A with $m(A) < \delta$, and any positive integer N, there is a positive integer $n \geq N$ and x not is A with $|g_n(x) - 0| \geq \varepsilon$. Yet the sequence, $\langle g_n \rangle$, does converge to zero point-wise.