4-14. a. Show that under the hypothesis of Theorem 4-17 we have $\int|f_n - f| \to 0$.

b. Let $<f_n>$ be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then $\int|f_n - f| \to 0$ if, and only if $\int|f_n| \to \int|f|$.

Proof. (a) If the hypotheses of Theorem 4-17 are satisfied, then there is a sequence $<g_n>$ of integrable functions that converge a.e. to an integrable function g with $|f_n| \le g_n$ for each n and the sequence of measurable functions $<f_n>$ converges a.e. to f. Suppose now that we have $\int g_n \to \int g$. Observe that $||f_n| - |f|| \le |f_n - f|$ at least almost everywhere. Thus,

$$\begin{aligned} |f_n - f| &\le |f_n| + |f| \\ &\le g_n + g \end{aligned}$$

a.e., since $|f_n| \to |f|$ a.e. , $|f_n| \le g_n$, and the sequence $<g_n>$ of integrable functions converge a.e. to the integrable function g. By using Proposition 4-15, it follows that the each of the functions of the sequence of functions $<h_n>$, where $h_n = |f_n - f|$, is integrable, $h_n \to 0$ a.e, and $\int g_n+g \to \int 2g$. Thus, applying Theorem 4-17 with the sequence $<h_n>$ in the role of the sequence $<f_n>$, the zero function in the role of f, the sequence $<g_n+g>$ in the role of $<g_n>$, and the function 2g in the place of g, it follows that $\int|f_n - f| \to 0$.

(b) Suppose now $<f_n>$ is a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. From Problem 4-10 it follows that $|f|$ is integrable and each of the funtions $|f_n|$ is integrable. Thus, if $\int|f_n| \to \int|f|$, we may apply part (a), since $g = |f|$ and the sequence $<g_n>$ with $g_n = |f_n|$ will allow us to satisfy the hypotheses of Theorem 4-17. Consequently, we have $\int|f_n - f| \to 0$. On the other hand, if $\int|f_n - f| \to 0$, we may use Proposition 4-15, Problem 4-10(a), and the inequality $||f_n| - |f|| \le |f_n - f|$ to infer that

$$\left|\int|f_n| - \int|f|\right| \le \int||f_n| - |f|| \le \int|f_n - f|.$$

Thus, $\int|f_n - f| \to 0$ implies that $\int|f_n| \to \int|f|$.