4-15. a. Let f be integrable over E. Then, given $\varepsilon > 0$, there is a simple function φ such that

$$\int_E |f - \varphi| < \varepsilon.$$

[Apply Problem 4-4 to the positive and negative parts of f.]

b. Under the same hypothesis there is a step function ψ such that

$$\int_E |f - \psi| < \varepsilon.$$

c. Under the same hypothesis there is a continuous function g vanishing outside a finite interval such that

$$\int_E |f - g| < \varepsilon.$$

Proof. (a) Let $\varepsilon > 0$. Decompose f into its positive and negative parts so that $f = f^+ - f^-$ with both f^+ and f^- nonnegative. From Problem 4-4 there are two increasing sequences of nonnegative simple functions $<\varphi_n>$ and $<\psi_n>$ with each function nonzero on a set of finite measure , $\lim \varphi_n = f^+$, and $\lim \psi_n = f^-$. By applying the Monotone Convergence Theorem and using the hypothesis that f is integrable, it follows that there is a positive integer so that $\int f^+ - \int \varphi_n < \varepsilon/2$ and $\int f^- - \int \psi_n < \varepsilon/2$. Set $\varphi = \varphi_n - \psi_n$. Then, since $|f - \varphi| \le |f^+ - \varphi_n| + |f^- - \psi_n|$, (a) follows by applying Proposition 4-15 and Exercise 4-10.//

(b) By applying Proposition 4-15, Exercise 4-10, and the triangle inequality for absolute values, we may assume f is an integrable simple function that is nonzero on a set of finite measure E. Unfortunately, E need not be an interval, and thus, we shall have to work a little to apply Proposition 3-22. Let $E_n = E \cap [-n,n]$ for each $n \in \mathbb{N}$, and let f_n be the function defined by $f_n = f \cdot \chi_n$, where χ_n is the characteristic function of E_n. Observe that we have $\lim f_n = f$ and $|f_n| \le |f|$, which is integrable on E since f is. Thus we may apply the Lebesgue Convergence Theorem to the sequence of functions $<|f - f_n|>$ to see that there is a positive integer n with $\int |f - f_n| < \varepsilon/2$. Now f_n is a measurable function that we may view as living on the set $[-n,n]$ by abusing notation a little. [If we were to be quite proper about this, we should distinguish between f_n and its restriction to the interval. We won't do so, however.] Plainly, since $|f|$ is integrable, the set where f_n assumes the values $\pm\infty$ must be of measure zero. In fact, since f_n is simple, there must be a number $K > 0$ such that $|f_n| \le K$ a.e. Applying Proposition 3-22 to f_n, it follows that there is a step function, ψ with $|f_n - \psi| < \delta$ except for a subset B of $[-n,n]$ of measure less than δ, where $\delta = \min(\varepsilon/(8K), \varepsilon/(8n))$ with the step function bounded by the same constant K. Since we have $\int_{[-n,n]}|f_n - \psi| \leq \int_B|f_n - \psi| + \int_{[-n,n]\sim B}|f_n - \psi| < \delta\cdot 2K + 2n\cdot\delta < \varepsilon/2$, all the pieces are in place to apply Proposition 4-15, Exercise 4-10, and the triangle inequality, and we are finished.

(c) Again applying Proposition 4-15, Exercise 4-10, and the triangle inequality for absolute values, we may assume f is a step function that is nonzero on an interval [a,b]. Now things are much easier than in part (b) since f is assumed to live on an interval. Simply apply Proposition 3-22 with ε replaced by $\delta = \min(\varepsilon/4K, \varepsilon/(2b-2a))$, where f is bounded in magnitude by K. Then there is a continuous function g defined on [a,b] that is bounded by the same constant K as f with $|f - g| < \delta$ except on a subset B of [a,b] of measure less than δ. Consequently, we have $\int_{[a,b]}|f - g| \leq \int_B|f - g| + \int_{[a,b]\sim B}|f - g| < \delta\cdot 2K + (b-a)\cdot\delta < \varepsilon$ and we are done.