4-4. Let f be a nonnegative measurable function.

a. Show that there is an increasing sequence $<\phi_{\rm n}>$ of nonnegative simple functions each of which vanishes outside of a set of finite measure such that f = lim $\phi_{\rm n}$.

b. Show that $\int f = \sup \int \phi$ over all simple functions $\phi \leq f$.

[It appears that (b) is slightly misstated. Either the simple functions may be arbitrary and nonzero on sets of finite measure **or** they may be nonnegative and defined on arbitrary measurable sets if the integrals of the "simple functions" are to be defined!! Probably Royden intended that the simple functions would be understood to be nonnegative. This seems to be the most reasonable interpretation under the circumstances, I think .]

Proof. (a) Rather than attempt to bare-hand this by constructing the desired sequence using the magical dyadic rationals perhaps, we shall "pour out the water" and use some of Royden's fine work to infer the existence of the sequence of varmints.

First, let f be a nonnegative measurable function. Then, for each n ϵ N, set $f_n = \min(f,n)\cdot\chi_n$, where χ_n is the characteristic function of the intersection of the set [-n,n] with the domain of f. Then each f_n is a bounded measurable function that is nonzero on a set of finite measure with $f_n \leq f$. Evidently, lim $f_n = f$ pointwise.

Next, recall that the first part of the proof of Proposition 4-3 provides us a sequence of simple functions that converges uniformly to any bounded measurable function living on a set of finite measure with each member of the sequence less than or equal to the given function.

We now put this to work. For each positive integer n, there is a simple function s_n with $s_n \leq f_n$, zero outside of the intersection of [-n,n] with the domain of f, and satisfying $|s_n(x) - f_n(x)| < n^{-1}$ for every x in the intersection of the domain of f and [-n,n]. Evidently, $s_n \leq f$ for each n εN , and lim $s_n = f$ pointwise on the domain of f.

The only problem that may remain is the sequence of simple functions may not be increasing and may not be nonnegative. To remedy these problems, it suffices to define the sequence $\langle \phi_n \rangle$ recursively as follows: Set $\phi_1 = \max(\phi_1, 0)$, and for n > 1, let $\phi_n = \max(\phi_{n-1}, s_n)$. Then it follows by induction that the sequence $\langle \phi_n \rangle$ will be increasing with each ϕ_n nonnegative, simple, and nonzero on a set of finite measure, and $s_n \leq \phi_n \leq f$. Furthermore, by means of a simple squeeze, $f = \lim \phi_n$. [Note: Take a look at the proof of Theorem 3-20.]//

(b) To see that $\int f = \sup \int \phi$ over all simple (nonnegative?) functions $\phi \leq f$, it suffices to show that $\int f \leq \sup \int \phi$ because Propositions 4-8 and $\phi \leq f$, imply that $\int f \geq \sup \int \phi$. This follows easily from part (a) using the Monotone Convergence Theorem, however, since the existence of the sequence $\langle \phi_n \rangle$ from part (a) then yields $\int f = \lim \int \phi_n = \sup \int \phi_n \leq \sup \int \phi$.//