4-6. Let $<f_n>$ be a sequence of nonnegative measurable functions that converge to f, and suppose that $f_n \le f$ for each n. Then

$$\int f = \lim \int f_n .$$

Proof. Fatou's Lemma implies that

$$\int f \le \liminf \int f_n .$$

The Proposition 4-8 and the inequality $f_n \le f$ imply that

$$\int f \ge \limsup \int f_n .$$

Apply Exercise 2-15.//

4-7. a. Show that we may have strict inequality in Fatou's Lemma.

Proof. Let the sequence of functions $<f_n>$ be defined on the real line by $f_n(x) = 1$ if $n \le x < n+1$, and $f_n(x) = 0$ otherwise. Evidently each function of the sequence is a nonnegative, simple, measurable, and equal to one on a set of measure equal to one. Thus, $\int f_n = 1$ for each n. Observe that $\lim f_n = 0$ pointwise. Thus, if $f = 0$, we have $0 = \int f < \liminf \int f_n = 1$.//

b. Show that the Monotone Convergence Theorem need not hold for decreasing sequences of functions.

Proof. Let the sequence of functions $<f_n>$ be defined on the real line by $f_n(x) = 0$ if $x < n$, and $f_n(x) = 1$ for $x \ge n$. Then each function of the sequence is nonnegative, simple, and measurable. Observe that $\lim f_n = 0$ pointwise. By applying the Monotone Convergence Theorem to an appropriate sequence of simple measurable functions $<\varphi_j>$ that increase to f_n and are 1 on a set with measure j --- the obvious varmints will do ---, it is easy to see that $\int f_n = \infty$ for each n. Thus, if f is the zero function, we have $0 = \int f < \lim \int f_n = \infty$.//