
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals", " $\Rightarrow$ " denotes "implies", and " $\Leftrightarrow$ " denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

1. (10 pts.) (a) Using implicit differentiation, compute dy/dx and d^2y/dx^2 when $x^2 + y^2 = 25$. **Label your expressions correctly or else.**

$d(x^2 + y^2)/dx = d(25)/dx$ implies that $dy/dx = -x/y$. Consequently, after differentiating one more time using quotient rule, replacing the occurrence of dy/dx in the expression for the second derivative, and cleaning up the algebra, we obtain $d^2y/dx^2 = -[y^2 + x^2]/y^3$.

(b) Obtain an equation for the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(1, -(24)^{1/2})$.

Evaluating the implicit derivative above at $(1, -(24)^{1/2})$ provides us with the slope of the tangent line, namely $1/(24)^{1/2}$. Consequently, an equation for the tangent line at the desired point is $y - (-(24)^{1/2}) = 1/(24)^{1/2}(x - 1)$.

2. (5 pts.) A 5-ft. ladder is leaning against the wall. If the top of the ladder slips down the wall at a rate of 2 ft./sec., how fast will the foot be moving away from the wall when the top is 3 ft. above the ground? [Try 4.6: 12-15 1st!]

Let $x(t)$ denote the distance from the foot of the ladder to the base of the wall, and let $y(t)$ denote the vertical distance from the top of the ladder to the ground. From the Pythagorean Theorem, we have

$$(*) \quad x^2(t) + y^2(t) = 5^2.$$

If t_0 denotes the time when the y is 3 ft., what we want is the value of $x'(t_0)$. Now $y(t_0) = 3$, $(*)$ above, and ' x ' being a distance imply that $x(t_0) = 4$. An implicit differentiation and a little obvious algebraic magic yield

$$x'(t_0) = -y(t_0)y'(t_0)/x(t_0).$$

Since $y'(t) = -2$ ft./sec., $x'(t_0) = (3)(2)/4 = 3/2$ feet per second. [Note: $y'(t) < 0$ since the distance from the top of the ladder to the ground is decreasing!]

3. (5 pts.) Use logarithmic differentiation to find dy/dx when $y = x^{\sin(x)}$. **Label your expressions correctly or else.**

Since $\ln(y) = \sin(x)\ln(x)$, by doing an implicit differentiation and transposing y , we get

$$y' = [\cos(x)\ln(x) + \sin(x) \cdot (1/x)]x^{\sin(x)}$$

4. (15 pts.) Differentiate the following functions. Do not attempt to simplify the algebra.

(a) $f(x) = \ln(3x^3 - x) - 2 \cdot \exp(4x^3 - 14)$

$$f'(x) = (9x^2 - 1)/(3x^3 - x) - 2 \cdot (12x^2) \exp(4x^3 - 14)$$

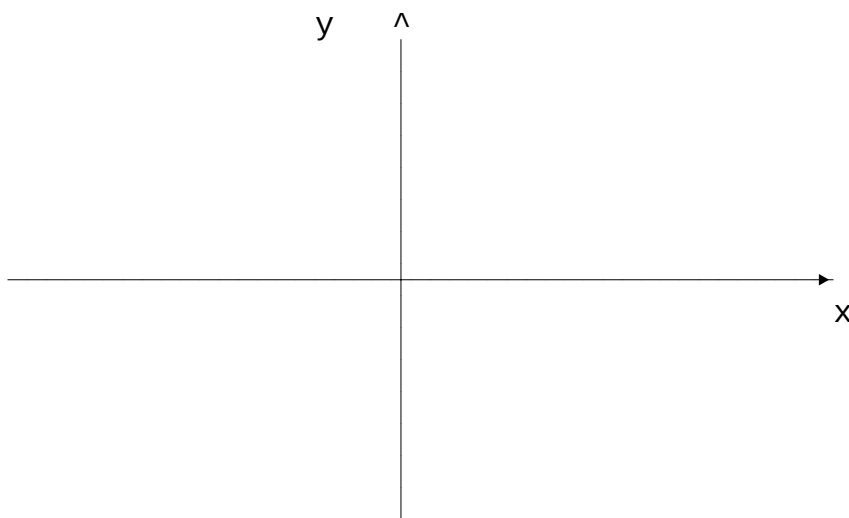
(b) $g(x) = 4^x + x^4 + 4^4 + \log_4(x) + \ln(4)$

$$g'(x) = \ln(4)4^x + 4x^3 + 0 + 1/(x \cdot \ln(4)) + 0$$

(c) $h(x) = \sec^{-1}(3x) + e^x \cdot \tan^{-1}(x) - 6 \cdot \cos^{-1}(x^2)$

$$h'(x) = 3/[|3x|((3x)^2 - 1)^{1/2}] + e^x \cdot \tan^{-1}(x) + e^x(1+x^2)^{-1} + 12x(1-x^4)^{-1/2}$$

5. (5 pts.) Carefully sketch the graph of $y = \tan^{-1}(x)$. Label very carefully. [*This may be viewed at [c1-t3m-g.htm](#).*]



6. (5 pts.) Solve for x without using a calculator. Use the natural logarithm when logarithms are needed.

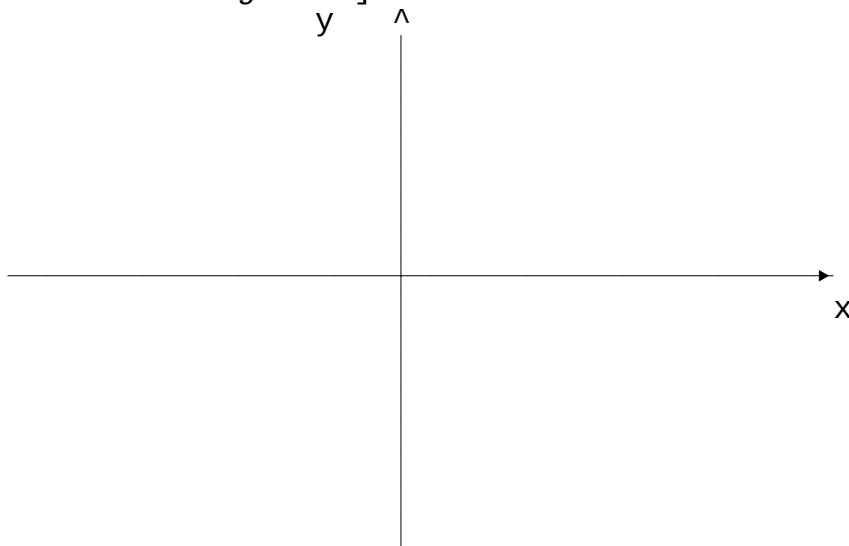
$$e^{2x} - 6e^x = -8$$

$$\begin{aligned} e^{2x} - 6e^x = -8 &\Leftrightarrow e^{2x} - 6e^x + 8 = 0 \\ &\Leftrightarrow (e^x - 4)(e^x - 2) = 0 \\ &\Leftrightarrow e^x - 4 = 0 \quad \text{or} \quad e^x - 2 = 0 \\ &\Leftrightarrow e^x = 4 \quad \text{or} \quad e^x = 2 \\ &\Leftrightarrow x = \ln(4) \quad \text{or} \quad x = \ln(2) \end{aligned}$$

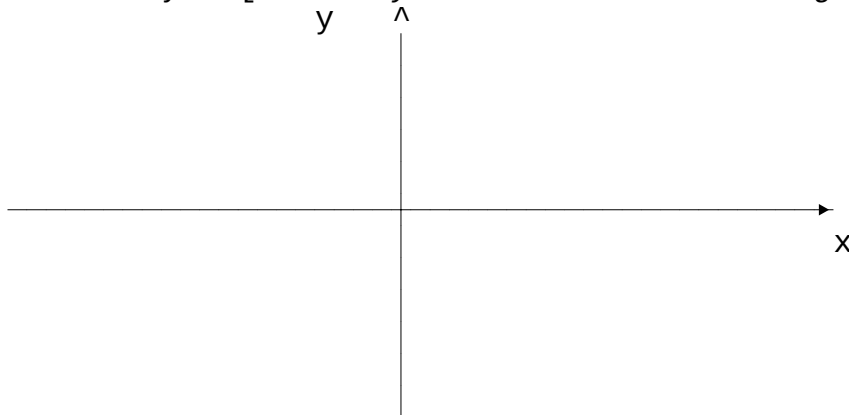
7. (5 pts.) Using a complete sentence and appropriate notation, provide the precise mathematical definitions for the following term: // The differential, dy , of a function $f(x)$ //

If f is differentiable, dy , the differential, is defined by the equation $dy = f'(x)dx$, where dx is treated as an independent variable.

8. (5 pts.) Carefully sketch both $f(x) = \ln(x)$ and $g(x) = e^x$ on the coordinate system below. **Label very carefully.** [This may be viewed at [c1-t3m-g.htm](#).]



9. (5 pts.) Carefully sketch the graph of $y = \sin^{-1}(x)$. Label very carefully. [This may be viewed at [c1-t3m-g.htm](#).]



10. (10 pts.) Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible.

$$(a) \lim_{t \rightarrow \infty} 6t \cdot \tan(3 \cdot t^{-1}) = 18$$

after writing

$$6t \cdot \tan(3 \cdot t^{-1}) = \frac{6 \cdot \tan(3 \cdot t^{-1})}{t^{-1}},$$

applying L'Hopital's Rule for $0/0$ forms once, and cancelling the obvious common factor from the numerator and denominator.

$$(b) \lim_{x \rightarrow 0} \frac{1 - \cos(3\pi x)}{e^x + e^{-x} - 2} = 9\pi^2/2$$

by applying L'Hopital's Rule for $0/0$ forms twice.

11. (5 pts.) The side of a cube is measured to be 30 feet with a possible error of ± 0.5 feet. Use differentials to estimate the relative error in the computed volume.

Set $V(x) = x^3$. The relative error in the computed volume is $\Delta V/V$. We shall approximate this using the differential dV . Thus, we have

$$\Delta V/V \approx dV/V = 3x^2 \cdot dx/x^3 \approx (3/x) \cdot \Delta x = (1/10) \cdot (\pm 0.5) = \pm 0.05$$

when $x = 30$ and $\Delta x = \pm 0.5$

12. (5 pts.) Find the exact value of $\cos[2 \cdot \cos^{-1}(3/5)]$.
[Warning: You will have to use some identities to handle this.]

$$\begin{aligned} \cos[2 \cdot \cos^{-1}(3/5)] &= \cos^2[\cos^{-1}(3/5)] - \sin^2[\cos^{-1}(3/5)] \\ &= 2 \cdot \cos^2[\cos^{-1}(3/5)] - 1 \\ &= 2 \cdot (3/5)^2 - 1 = (18 - 25)/25 = -7/25 \end{aligned}$$

13. (5 pts.) Let $f(x) = 4x^3 + 5x$. (a) Show f is invertible.
 (b) Then solve the equation $f^{-1}(x) = -2$.

(a) For each $x \in \mathbb{R}$ we have $f'(x) = 12x^2 + 5 \geq 5$. Thus $f'(x) > 0$ on $\mathbb{R} = (-\infty, \infty)$. It follows that f has an inverse.

(b) $f^{-1}(x) = -2 \quad \Leftrightarrow \quad x = f(-2)$
 $\Leftrightarrow \quad x = 4(-2)^3 + 5(-2)$
 $\Leftrightarrow \quad x = -42$

14. (5 pts.) Use differentials and a linear approximation formula to estimate $(24)^{1/2}$. [Hint: Use $x_0 = 25$ and $f(x) = x^{1/2}$.]

Let $x_0 = 25$. Then $x_0 + \Delta x = 24$ implies that $\Delta x = -1$.
 Consequently,

$$(24)^{1/2} = f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x = 5 - (1/10) \\ = 4.9 \text{ since } f'(x_0) = (1/2) \cdot (25)^{-1/2} = 1/10.$$

15. (5 pts.) Solve for x without using a calculator.

$$\ln(16x) - 2 \cdot \ln(x^2) = \ln(2)$$

Using standard properties of the natural log function, you can obtain $x = 8^{1/3} = 2$.

16. (5 pts.) Carefully sketch the graph of $y = \cos^{-1}(x)$. Label very carefully. [This may be viewed at [c1-t3m-g.htm](#).]

