Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (16 pts.) Fill in the blanks of the following analysis with the correct terminology.

Let $f(x) = x^4 - 4x^3$. Then $f'(x) = 4x^3 - 12x^2 = 4(x - 0)^2(x - 3)$.

Consequently, x = 0 and x = 3 are <u>critical (stationary)</u> points of f. Since $f'(x) > 0$ for $3 < x$, f is <u>increasing</u> on the set $(3,\infty)$. Also, because $f'(x) < 0$ when $0 < x < 3$ or $x < 0$, and f is continuous, f is <u>decreasing</u> on the interval $(-\infty , 3)$. Using the first derivative test, it follows that f has a(n) <u>relative minimum</u> at x = 3, and <u>neither a relative max nor a relative min</u> at x = 0.

Since $f''(x) = 12x^2 - 24x = 12x\cdot(x - 2)$, we have $f''(0) = 0$, $f''(2) = 0$, $f''(x) < 0$ when $0 < x < 2$, and $f''(x) > 0$ when $x > 2$ or $x < 0$. Thus, f is <u>concave down</u> on the interval $(0,2)$, f is <u>concave up</u> on the set $(-\infty ,0) \cup (2,\infty)$, and f has <u>inflection points</u> at x = 0 and x = 2.

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2. (4 pts.) Rolle's Theorem states that if f(x) is continuous on [a,b] with f(a) = f(b) = 0 and differentiable on (a,b), then there is a number c in (a,b) such that f′(c) = 0. **Give an example of a function f(x) defined on [-1,1] with f differentiable on (-1,1) and f(-1) = f(1) = 0 but such that there is no number c in (-1,1) with f′(c) = 0.** [Hint: Which hypothesis above must you violate??]

A suitable example clearly must fail to be continuous on the interval [-1,1] and yet satisfy the remaining hypotheses. Here is one such, defined in pieces, of course.

$$f(x) = \begin{cases} 0 & \text{if } x = -1 \text{ or } x = 1 \\ x & \text{if } -1 < x < 1 \end{cases}$$

Then f(-1) = f(1) = 0, and f′(x) = 1 for every x in the interval (-1,1). Hence there is no point c in (-1,1) where f′(c) = 0.

3. (10 pts.) Find all the critical points of the function $f(x) = 3\cdot(x^2 + 6x)^{1/3}$. Which critical points are stationary points? Apply the second derivative test at each stationary point and draw an appropriate conclusion.

First, $f'(x) = (2x + 6)/(x^2 + 6x)^{2/3} = 2(x + 3)/(x(x + 6))^{2/3}$ for $x \neq 0$ and $x \neq -6$. Therefore, f has critical points at $x = -6$, $x = -3$, and $x = 0$. Only $x = -3$ is a stationary point. Observe that we have

$$f''(x) = \frac{[2(x^2 + 6x)^{2/3} - (2x + 6)(2/3)(x^2 + 6x)^{-1/3}(2x + 6)]}{(x^2 + 6x)^{4/3}}$$

for $x \neq 0$ and $x \neq -6$. It follows that $f''(-3) > 0$. Consequently, the second derivative test implies that f has a relative minimum at $x = -3$. [To see $f''(-3) > 0$ easily, note that if $x = -3$, the second term of the numerator of f'' is zero, and what remains is a quotient of squares.]

4. (10 pts.) Locate and determine the maximum and minimum values of the function $f(x) = 2x^3 - 6x^2$ on the interval [1,4]. What magic theorem allows you to conclude that f(x) has a maximum and minimum even before you attempt to locate them? Why??

Since f is a polynomial, f is continuous on the interval [1,4]. Consequently, the magical Extreme Value Theorem guarantees that f has absolute extrema of both flavors on [1,4]. Since we have $f'(x) = 6x^2 - 12x = 6x(x - 2)$, f has only one critical point in (1,4), namely $x = 2$. To finish this, it suffices to evaluate f at the critical point and on the boundary of the set [1,4] in order to pick out the extrema. Since $f(1) = -4$, $f(2) = -8$, and $f(4) = 128 - 96 = 32$, the maximum is 32 and occurs at $x = 4$, and the minimum is -8 and is located at $x = 2$.

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5. (10 pts.) (a) State the Mean Value Theorem of Differential Calculus. Use a complete sentence and appropriate notation.
// If f is continuous on the closed interval [a,b] and differentiable on the open interval (a,b), then there is at least one number c in the interval (a,b) with

$$f'(c) = [f(b) - f(a)]/[b - a].$$

(b) Show how to use the Mean Value Theorem to prove the following: If x and y are real numbers, then

$$(*) \qquad |\sin(3x) - \sin(3y)| \le 3|x - y|$$

is true. // First, if $x = y$, then both sides of (*) are zero, and thus the inequality is trivially true. Consequently, it suffices to handle the case where $x \ne y$, where we may assume $x < y$ without any loss of generality. Thus pretend $x < y$. Only a little thought reveals that $f(t) = \sin(3t)$ is continuous on the closed interval [x,y] and differentiable on the interval (x,y). From the Mean Value Theorem, there must be at least one number c in the interval (x,y) with

$$f'(c) = [f(y) - f(x)]/[y - x].$$

Since $f'(t) = 3\cdot\cos(3t)$ and $|\cos(3t)| \le 1$, we have $|\sin(3y) - \sin(3x)|/|y - x| = 3|\cos(3c)| \le 3$. This, of course, is equivalent to (*).

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6. (10 pts.) A rectangular area of 100 square feet is to be fenced. Three of the sides will use fencing costing $2.00 per running foot, and the remaining side will use a hedge costing $1.00 per running foot. Find the dimensions of the rectangle which has the least cost to enclose. Provide a complete enough analysis to convince the doubtful that your extreme value is an absolute minimum.// Denote the lengths of the perpendicular sides of the rectangle by "x" and "y". For definiteness, let one of the "y" sides and both of the "x" sides cost $2 per foot, and let the remaining "y" side cost $1 per foot. [You could, of course, do this the other way around!!] Then $xy = 100$, and the cost in terms of the dimensions is $C = y + 2y + 2x + 2x$ dollars. If we write this in terms of x alone, then we have $C(x) = 3(100/x) + 4x = 4x + 300x^{-1}$ for $x > 0$. [Neither x nor y may be zero. Why?] Now

$$C'(x) = 4 - 300x^{-2}$$

$$= (4/x^2)\cdot(x^2 - 75)$$

$$= (4/x^2)\cdot(x - (75)^{1/2})\cdot(x + (75)^{1/2})$$

for $x > 0$. Consequently, the only critical point is $x = (75)^{1/2}$. Since we have $C''(x) = 600x^{-3}$, it follows that $C''((75)^{1/2}) > 0$. The Second Derivative Test implies that $C((75)^{1/2})$ is a relative minimum. Since this is the only minimum on $(0,\infty)$ and C(x) is continuous, $C((75)^{1/2})$ is an absolute minimum. The dimensions: $x = (75)^{1/2} = 5(3)^{1/2}$ and $y = (100/(75)^{1/2}) = 20/(3)^{1/2}$ feet.

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7. (5 pts.) Find the function f(x) that satisfies the following two equations: $f'(x) = \cos(x) + \sin(x)$ for all x and $f(\pi) = 4$.

Since $f'(x) = \cos(x) + \sin(x)$, it follows that

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int \cos(x) + \sin(x)\,dx \\ &= \sin(x) - \cos(x) + c \end{aligned}$$

for some real number c. From what we now know about the structure of f, $f(\pi) = 4$ implies that $4 = \sin(\pi) - \cos(\pi) + c$. Solving this little linear equation yields $c = 3$. Thus, $f(x) = \sin(x) - \cos(x) + 3$.

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8. (5 pts.) Find a function g(x) so that f satisfies the following equation:

$$\int g(x)dx = \sec^2(x) - 2x + 4e^x + C$$

From the very definition of antiderivative and how we have agreed to denote such a varmint, it follows that g must satisfy the equation

$$\begin{aligned} g(x) &= (\sec^2(x) - 2x + 4e^x + C)' \\ &= 2\cdot\sec(x)\cdot\sec(x)\cdot\tan(x) - 2 + 4e^x \end{aligned}$$

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9. (10 pts.) Evaluate the following antiderivatives.

$$\int 5x^4 - \frac{4}{x} - \frac{24}{x^2} - \frac{7}{|x|(x^2 - 1)^{1/2}}\,dx =$$

$$x^5 - 4\cdot\ln|x| + 24x^{-1} - 7\cdot\sec^{-1}(x) + C$$

or

$$x^5 - 4\cdot\ln|x| + 24x^{-1} + 7\cdot\csc^{-1}(x) + C$$

$$\int e^{4x-6} - \frac{10\cdot\cos(x)}{\sin^2(x) + 1}\,dx = (1/4)e^{4x-6} - 10\tan^{-1}(\sin(x)) + C$$

Break this integral into a difference of integrals. The first can be handled with the substitution $u = 4x - 6$ and the second can be done using the substitution $v = \sin(x)$.

10\. (20 pts.) Very carefully sketch each of the following functions. Label very carefully.

(a) $f(x)$ is continuous on $\mathbb{R}$ and satisfies the following:

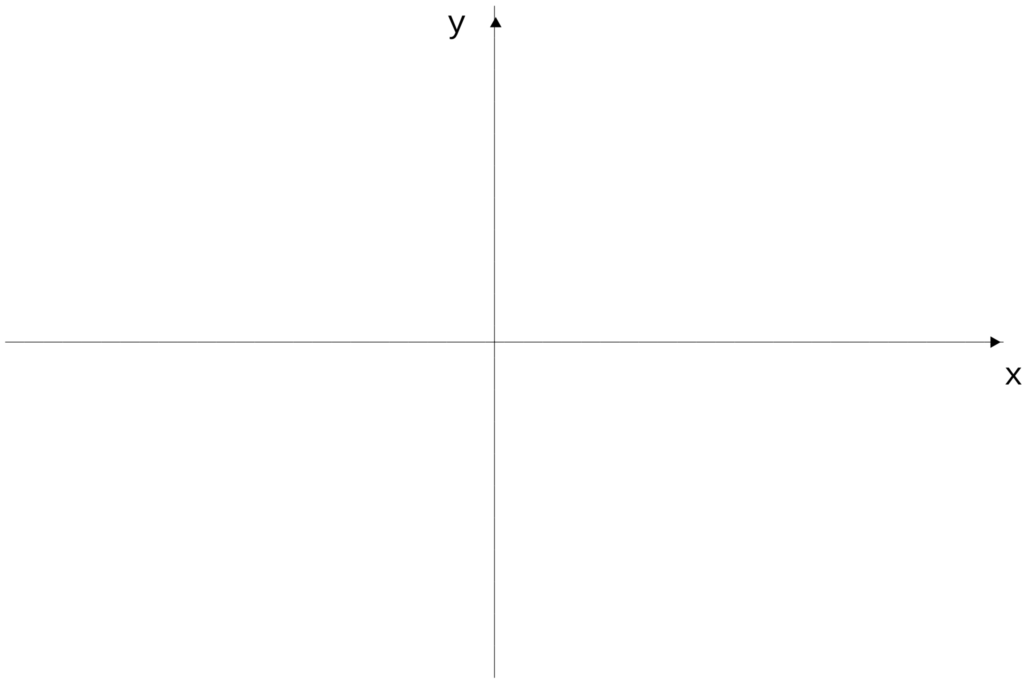
(1) $f(0) = 0$;

(2) $x < 0 \Rightarrow f'(x) < 0$, and $x > 0 \Rightarrow f'(x) > 0$;

(3) $\lim_{x \to 0^-} f'(x) = -\infty$ and $\lim_{x \to 0^+} f'(x) = +\infty$;

(4) $x \neq 0 \Rightarrow f''(x) < 0$; and

(5) $f(x) \longrightarrow 2$ as $x \longrightarrow \pm\infty$. **See c1-t4m-g.htm.**



(b) $g(x) = x/(x^2 + 4)$ [Analyze g' and g'' and how g behaves as $x \longrightarrow \pm\infty$. Work on the back of page 4!] **See c1-t4m-g.htm.**