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**Read Me First:** Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , " $\Rightarrow$ " denotes "implies" , and " $\Leftrightarrow$ " denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.

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1. (20 pts.) Compute the derivatives of the following functions. You may use any of the rules of differentiation that are at your disposal. Do not attempt to simplify the algebra in your answers.

(a)  $f(x) = 4x^5 - 3x^{-6} + 7\tan(x)$

$$f'(x) = 20x^4 + 18x^{-7} + 7\sec^2(x)$$

(b)  $g(x) = (x^{-3} - 4x)\cos(x)$

$$g'(x) = (-3x^{-4} - 4)\cos(x) - (x^{-3} - 4x)\sin(x)$$

(c)  $h(t) = \frac{3\sec(t)}{2t^5 - \cot(t)}$

$$h'(t) = \frac{3\sec(t)\tan(t)(2t^5 - \cot(t)) - (3\sec(t)(10t^4 + \csc^2(t)))}{(2t^5 - \cot(t))^2}$$

(d)  $L(x) = 3 \cdot \tan^2(4x^3)$

$$L'(x) = 6 \cdot \tan(4x^3) \cdot \sec^2(4x^3) \cdot 12x^2$$

(e)  $y = \sin(10x) + 5\cos(3x) + \csc(\pi/4)$

$$\frac{dy}{dx} = 10\cos(10x) - 15\sin(3x)$$

Note:  $\csc(\pi/4)$  is a constant.

2. (10 pts.) (a) Using complete sentences and appropriate notation, state the Intermediate Value Theorem.

If  $f$  is continuous on a closed interval  $[a,b]$ , and  $k$  is any number between  $f(a)$  and  $f(b)$ , inclusive, then there is a number  $x_0$  in the interval  $[a,b]$  with  $f(x_0) = k$ .

(b) Can the Intermediate Value Theorem be used to show the equation

$$(x + 1)/(x^2 - 4x - 5) = 0$$

has a solution in the interval  $[-2,6]$ ? Explain completely.

[Hint: Let  $f(x) = (x + 1)/(x^2 - 4x - 5)$  on  $[-2,6]$ . Discuss appropriate properties of  $f$  and then draw a suitable conclusion.]

Let  $f(x) = (x + 1)/(x^2 - 4x - 5)$  on  $[-2,6]$ . Observe that  $f(-2) = -1/7$ ,  $f(6) = 1$ , and  $k = 0$  is a number between  $f(-2)$  and  $f(6)$ . Unfortunately,  $f$  is not continuous on the interval  $[-2,6]$ . Thus, we cannot use the Intermediate Value Theorem to imply there is a number  $x_0$  in  $[-2,6]$  where  $f(x_0) = 0$ . [In fact, it is easy to see that  $f$  is never zero since  $f(x) = 1/(x-5)$  for  $x$  different from  $-1$  and  $5$ .]

3. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative,  $f'(x)$ , of a function  $f(x)$ .

The function  $f'$  defined by the equation

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

is called the derivative of  $f$  with respect to  $x$ . The domain of  $f'$  consists of all  $x$  in the domain of  $f$  for which the limit above exists.

(b) Using the definition of the derivative as a limit, show all steps of the computation of  $f'(x)$  when  $f(x) = 4x^2 - x$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[4(x+h)^2 - (x+h)] - [4x^2 - x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{8xh + 4h^2 - h}{h} \\ &= \lim_{h \rightarrow 0} (8x - 1 + 4h) \\ &= 8x - 1 \end{aligned}$$

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4. (10 pts.) Using complete sentences and appropriate notation, provide the precise mathematical definitions for each of the following items:

(a)  $\lim_{x \rightarrow a} f(x) = L$  [Hint: This involves  $\varepsilon$  and  $\delta$ .] //

Suppose that  $f$  is a function that is defined everywhere in some open interval containing  $x = a$ , except possibly at  $x = a$ . We write

$$\lim_{x \rightarrow a} f(x) = L$$

if  $L$  is a number such that for each  $\varepsilon > 0$  we can find a  $\delta > 0$ , such that if  $x$  is in the domain of  $f$  and  $0 < |x - a| < \delta$ , then  $|f(x) - L| < \varepsilon$ .

(b) **Continuity** of a function  $f(x)$  at a point  $x = a$  //

A function  $f$  is continuous at  $x = a$  if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

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5. (5 pts.) Find a value for the constant  $k$ , if possible, that will make the function  $h(x)$  defined below continuous at  $x = -1$ . If you find such a  $k$ , using the definition, verify the continuity of  $h(x)$  at  $x = -1$ . Suppose that

$$h(x) = \begin{cases} (x + 1)/(x^2 - 4x - 5) & , \quad x \neq -1 \\ 10 \cdot k & , \quad x = -1. \end{cases}$$

In order for  $h$  to be continuous at  $x = -1$ , it is necessary and sufficient for

$$\begin{aligned} 10 \cdot k = h(-1) &= \lim_{x \rightarrow -1} h(x) = \lim_{x \rightarrow -1} (x + 1)/(x^2 - 4x - 5) = \lim_{x \rightarrow -1} 1/(x - 5) \\ &= -1/6. \quad \text{Thus, } k = -1/60. \end{aligned}$$

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6. (5 pts.) Evaluate the following limit. To obtain full credit, you must show all essential steps correctly in a chain of equations.

$$\lim_{\theta \rightarrow 0} \frac{\tan(8\theta)}{\sin(3\pi\theta)} = \dots = 8/3\pi$$

To get full credit, the intermediate steps must be provided, and they must be correct. The numerical answer by itself will not suffice here.

7. (5 pts.) Compute  $f''(x)$  when  $f(x) = \tan(3x^2)$ . Label your expressions correctly or else.

$$f'(x) = 6x \cdot \sec^2(3x^2) \quad [\text{Observe that } f' \text{ is a product!}]$$

$$\begin{aligned} f''(x) &= 6 \cdot \sec^2(3x^2) + (6x) \cdot 2 \cdot \sec(3x^2) \cdot \sec(3x^2) \cdot \tan(3x^2) \cdot (6x) \\ &= 6 \cdot \sec^2(3x^2) + (72x^2) \cdot \sec^2(3x^2) \cdot \tan(3x^2) \end{aligned}$$

8. (5 pts.) Obtain an equation for the line tangent to the graph of  $f(x) = \tan(x)$  at  $x_0 = \pi/6$ .

$$y - (3^{1/2}/3) = (4/3) \cdot (x - (\pi/6))$$

will do after the dust settles. [ $f'(x) = \sec^2(x)$  plus assorted trig treats are needed. For instance,  $\sec(x) = 1/\cos(x)$ .]

9. (5 pts.) Suppose  $f(x) = x^2 \sin(1/x^2)$  for  $x \neq 0$ . Show an evaluation of the following limit that is completely correct. You will need to build a suitable inequality to provide a complete solution.

$$\lim_{x \rightarrow 0} x^2 \sin(1/x^2) =$$

To see this, observe that for  $x \neq 0$ ,  $-1 \leq \sin(1/x^2) \leq 1$  implies that  $-x^2 \leq x^2 \sin(1/x^2) \leq x^2$ . Since  $x^2 \rightarrow 0$  as  $x \rightarrow 0$ , the squeezing theorem implies  $x^2 \sin(1/x^2) \rightarrow 0$  as  $x \rightarrow 0$ .

10. (5 pts.) Find all points in the interval  $[-2\pi, 2\pi]$  where the graph of  $f(x) = x - \cos(x)$  has a horizontal tangent line.

The graph of  $f$  has a horizontal tangent line where  $f'(x) = 0$ . Now  $f'(x) = 1 + \sin(x)$ . Thus  $f'(x) = 0$  in the given interval where  $\sin(x) = -1$ . This turns out to be at two points,  $x_0 = 3\pi/2$ , and  $x_1 = -\pi/2$ .

14. (5 pts.) Give an  $\varepsilon - \delta$  proof that  $\lim_{x \rightarrow 2} (8x - 2) = 14$ .

Proof: Let  $\varepsilon > 0$  be arbitrary. Set  $\delta = \varepsilon/8$ . Observe that  $\delta > 0$ . Suppose now that  $x$  satisfies  $0 < |x - 2| < \delta$ . We shall now verify that  $0 < |x - 2| < \delta$  implies  $|(8x - 2) - 14| < \varepsilon$ . Now

$$\begin{aligned} 0 < |x - 2| < \delta &\Rightarrow |x - 2| < \varepsilon/8 \\ &\Rightarrow 8|x - 2| < \varepsilon \\ &\Rightarrow |8x - 16| < \varepsilon \\ &\Rightarrow |(8x - 2) - 14| < \varepsilon. \end{aligned}$$

Since, given an arbitrary  $\varepsilon > 0$ , we have produced a number  $\delta > 0$  such that, if  $x$  satisfies  $0 < |x - 2| < \delta$ , then  $|(8x - 2) - 14| < \varepsilon$ , we have proved that  $(8x - 2) \rightarrow 14$  as  $x \rightarrow 2$ . [Without *scratching*.]

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11. (5 pts.) Obtain the following limit. This is easy if you grok the definition of the derivative.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\cos((\pi/3) + h) - \cos(\pi/3)}{h} &= (\cos)'(\pi/3) \\ &= -\sin(\pi/3) = -3^{1/2}/2\end{aligned}$$

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12. (5 pts.) Pretend  $f$  is a magical function that has the property that at  $x = -2$  the tangent line to the graph of  $f$  is actually defined by the equation  $y = -7x + 9$ . Obtain the following: (a)  $f'(-2) = -7$  (b)  $f(-2) = 23$

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13. (5 pts.) Is the function  ~~$h$~~   $g$  defined below differentiable at  $x = 0$  ?? Prove your guess is correct.

$$g(x) = \begin{cases} (x - 1)^2 & , \quad x \geq 0 \\ -2x + 4 & , \quad x < 0 \end{cases}$$

There are two natural ways to tackle this problem. You could simply try to use the definition of the derivative as a limit. In doing that, you would have to deal with the critical limit as two one-sided limits because of the way the function  $g$  is defined. Alternatively, you might be tempted to use the theorem on page 199 that deals with just this sort of situation. You should try solving this using the definition. Below, I'll show you how the funny theorem won't solve the problem.

$$\text{First, } g(0) = (0 - 1)^2 = 1,$$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-2x + 4) = 4, \text{ and}$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x - 1)^2 = 1.$$

Thus,  $g$  is not continuous at  $x = 0$ . This means that although

$$\lim_{x \rightarrow 0^-} g'(x) = \lim_{x \rightarrow 0^-} -2 = -2, \text{ and}$$

$$\lim_{x \rightarrow 0^+} g'(x) = \lim_{x \rightarrow 0^+} 2(x - 1) = -2,$$

the funny theorem doesn't imply that  $g'(0) = -2$ . We haven't satisfied all the hypotheses of the theorem. *The Solution:* From the work above dealing with the continuity of  $g$ ,  $g$  not continuous at  $x = 0$  implies  $g$  is not differentiable at  $x = 0$ .

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14. (5 pts.) Give an  $\epsilon - \delta$  proof that  $\lim_{x \rightarrow 2} (8x - 2) = 14$ .

[This proof appears on Page 4 below Problem 10.]