
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (10 pts.) (a) Using implicit differentiation, compute dy/dx and d^2y/dx^2 when $x^2 + y^2 = 16$. **Label your expressions correctly or else.**

$d(x^2 + y^2)/dx = d(16)/dx$ implies that $dy/dx = -x/y$. Consequently, after differentiating one more time using quotient rule, replacing the occurance of dy/dx in the expression for the second derivative, and cleaning up the algebra, we obtain $d^2y/dx^2 = -[y^2 + x^2]/y^3$.

(b) Obtain an equation for the line tangent to the graph of $x^2 + y^2 = 16$ at the point $(1,-(15)^{1/2})$.

Evaluating the implicit derivative above at $(1,-(15)^{1/2})$ provides us with the slope of the tangent line, namely $1/(15)^{1/2}$. Consequently, an equation for the tangent line at the desired point is $y -(-(15)^{1/2}) = 1/(15)^{1/2}(x - 1)$.

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2. (5 pts.) A 5-ft. ladder is leaning against the wall. If the top of the ladder slips down the wall at a rate of 2 ft./sec., how fast will the foot be moving away from the wall when the top is 4 ft. above the ground? [Try 4.6: 12-15 1st!]

Let x(t) denote the distance from the foot of the ladder to the base of the wall, and let y(t) denote the vertical distance from the top of the ladder to the ground. From the Pythagorean Theorem, we have

(*) $$x^2(t) + y^2(t) = 5^2.$$

If t_0 denotes the time when the y is 4 ft., what we want is the value of $x'(t_0)$. Now $y(t_0) = 4$, (*) above, and 'x' being a distance imply that $x(t_0) = 3$. An implicit differentiation and a little obvious algabraic magic yield

$$x'(t_0) = -y(t_0)y'(t_0)/x(t_0).$$

Since $y'(t) = -2$ ft./sec., $x'(t_0)= (4)(2)/3 = 8/3$ feet per second. [Note: $y'(t) < 0$ since the distance from the top of the ladder to the ground is decreasing!]

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3. (5 pts.) Use logarithmic differentiation to find dy/dx when $y = x^{\cos(x)}$. **Label your expressions correctly or else.**

Since $\ln(y) = \cos(x)\ln(x)$, by doing an implicit differentiation and transposing y, we get

$y' = [\cos(x)\cdot(1/x) - \sin(x)\ln(x)]x^{\cos(x)}$

4. (15 pts.) Differentiate the following functions. Do not attempt to simplify the algebra.

(a) $f(x) = \ln(3x^3 - 7x) - 2\cdot\exp(5x^3 - 14)$

$f'(x) = (9x^2-7)/(3x^3 - 7x) - 2\cdot(15x^2)\exp(5x^3 - 14)$

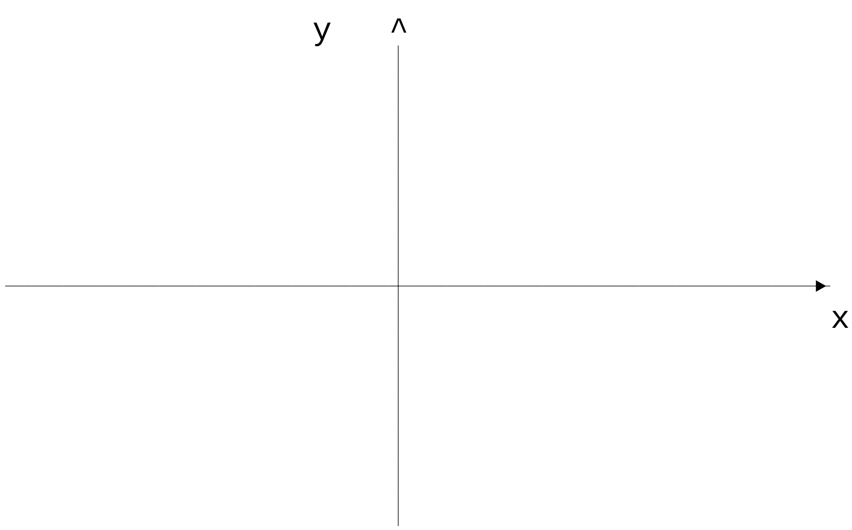
(b) $g(x) = 5^x + x^5 + 5^5 + \log_5(x) + \ln(5)$

$g'(x) = \ln(5)5^x + 5x^4 + 0 + 1/(x\cdot\ln(5)) + 0$

(c) $h(x) = \sec^{-1}(4x) + e^x\cdot\tan^{-1}(x) - 8\cdot\cos^{-1}(x^2)$

$h'(x) = 4/[|4x|((4x)^2-1)^{1/2}] + e^x\cdot\tan^{-1}(x)+e^x(1+x^2)^{-1}+ 16x(1-x^4)^{-1/2}$

5. (5 pts.) Carefully sketch the graph of $y = \tan^{-1}(x)$. Label very carefully. [*This may be viewed at c1-t3t-g.htm.*]



6. (5 pts.) Solve for x without using a calculator. Use the natural logarithm when logarithms are needed.

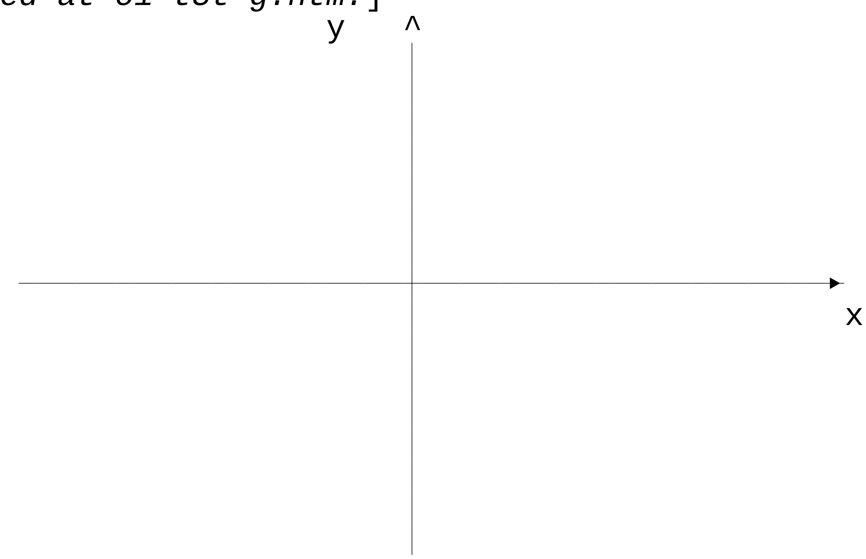
$$e^{2x} - 8e^{x} = -15$$

$$\begin{aligned} e^{2x} - 8e^{x} = -15 \quad &\Leftrightarrow \quad e^{2x} - 8e^{x} + 15 = 0 \\ &\Leftrightarrow \quad (e^{x} - 3)(e^{x} - 5) = 0 \\ &\Leftrightarrow \quad e^{x} - 3 = 0 \ \text{ or } \ e^{x} - 5 = 0 \\ &\Leftrightarrow \quad e^{x} = 3 \ \text{ or } \ e^{x} = 5 \\ &\Leftrightarrow \quad x = \ln(3) \text{ or } x = \ln(5) \end{aligned}$$

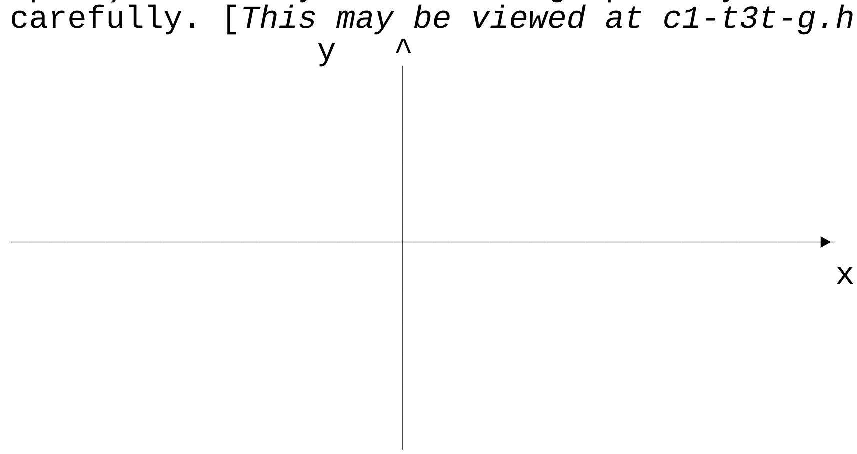
7. (5 pts.) Using a complete sentence and appropriate notation, provide the precise mathematical definitions for the following term: // The differential, dy, of a function f(x) //

If f is differentiable, dy, the differential, is defined by the equation dy = f′(x)dx, where dx is treated as an independent variable.

8. (5 pts.) Carefully sketch both f(x) = ln(x) and g(x) = e^{x} on the coordinate system below. **Label very carefully**. [*This may be viewed at c1-t3t-g.htm.*]



9. (5 pts.) Carefully sketch the graph of y = $\sin^{-1}(x)$. Label very carefully. [*This may be viewed at c1-t3t-g.htm.*]



10. (10 pts.) Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible.

(a) $\lim_{t \to \infty} 7t \cdot \tan(3 \cdot t^{-1}) = 21$

after writing

$$7t \cdot \tan(3 \cdot t^{-1}) = \frac{7 \cdot \tan(3 \cdot t^{-1})}{t^{-1}},$$

applying L'Hopital's Rule for 0/0 forms once, and cancelling the obvious common factor from the numerator and denominator.

(b) $\lim_{x \to 0} \frac{1 - \cos(5\pi x)}{e^{x} + e^{-x} - 2} = 25\pi^2/2$

by applying L'Hopital's Rule for 0/0 forms twice.

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11. (5 pts.) The side of a cube is measured to be 15 feet with a possible error of ±0.5 feet. Use differentials to estimate the relative error in the computed volume.

Set $V(x) = x^3$. The relative error in the computed volume is $\Delta V/V$. We shall approximate this using the differential dV. Thus, we have

$\Delta V/V \approx dV/V = 3x^2 \cdot dx/x^3 \approx (3/x) \cdot \Delta x = (1/5) \cdot (\pm 0.5) = \pm 0.1$

when $x = 15$ and $\Delta x = \pm 0.5$

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12. (5 pts.) Find the exact value of $\cos[2 \cdot \sin^{-1}(3/5)]$. [**Warning**: You will have to use some identities to handle this.]

$\cos[2 \cdot \sin^{-1}(3/5)] = \cos^2[\sin^{-1}(3/5)] - \sin^2[\sin^{-1}(3/5)]$

$= 1 - 2 \cdot \sin^2[\sin^{-1}(3/5)]$

$= 1 - 2 \cdot (3/5)^2 = (25 - 18)/25 = 7/25$

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13. (5 pts.) Let $f(x) = 5x^3 + 5x$. (a) Show f is invertible. (b) Then solve the equation $f^{-1}(x) = -2$.

(a) For each $x \varepsilon \mathbb{R}$ we have $f'(x) = 15x^2 + 5 \geq 5$. Thus $f'(x) > 0$ on $\mathbb{R} = (-\infty,\infty)$. It follows that f has an inverse.

(b) $f^{-1}(x) = -2 \quad \Leftrightarrow \quad x = f(-2)$
$\Leftrightarrow \quad x = 5(-2)^3 + 5(-2)$
$\Leftrightarrow \quad x = -50$

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14. (5 pts.) Use differentials and a linear approximation formula to estimate $(15)^{1/2}$. [Hint: Use $x_0 = 16$ and $f(x) = x^{1/2}$.]

Let $x_0 = 16$. Then $x_0 + \Delta x = 15$ implies that $\Delta x = -1$. Consequently,

$(15)^{1/2} \quad = f(x_0 + \Delta x) \quad \approx \quad f(x_0) + f'(x_0)\Delta x \quad = 4 - (1/8)$

$= 31/8$ since $f'(x_0) = (1/2)\cdot(16)^{-1/2} = 1/8$.

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15. (5 pts.) Solve for x without using a calculator.

$\ln(81x) - 2\cdot\ln(x^2) = \ln(3)$

Using standard properties of the natural log function, you can obtain $x = 27^{1/3} = 3$.

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16. (5 pts.) Carefully sketch the graph of $y = \cos^{-1}(x)$. Label very carefully. [*This may be viewed at c1-t3t-g.htm.*]