
Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: " $=$ " denotes "equals" , " \Rightarrow " denotes "implies" , and " \Leftrightarrow " denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.

1. (16 pts.) Fill in the blanks of the following analysis with the correct terminology.

Let $f(x) = x^4 - 8x^3$. Then $f'(x) = 4x^3 - 24x^2 = 4(x - 0)^2(x - 6)$.

Consequently, $x = 0$ and $x = 6$ are critical (stationary) points of f . Since $f'(x) > 0$ for $6 < x$, f is increasing on the set $(6, \infty)$. Also, because $f'(x) < 0$ when $0 < x < 6$ or $x < 0$, and f is continuous, f is decreasing on the interval $(-\infty, 6)$. Using the first derivative test, it follows that f has a(n) relative minimum at $x = 6$, and neither a relative max nor a relative min at $x = 0$.

Since $f''(x) = 12x^2 - 48x = 12x(x - 4)$, we have $f''(0) = 0$, $f''(4) = 0$, $f''(x) < 0$ when $0 < x < 4$, and $f''(x) > 0$ when $x > 4$ or $x < 0$. Thus, f is concave down on the interval $(0, 4)$, f is concave up on the set $(-\infty, 0) \cup (4, \infty)$, and f has inflection points at $x = 0$ and $x = 4$.

2. (4 pts.) Rolle's Theorem states that if $f(x)$ is continuous on $[a, b]$ with $f(a) = f(b) = 0$ and differentiable on (a, b) , then there is a number c in (a, b) such that $f'(c) = 0$. **Give an example of a function $f(x)$ defined on $[-1, 1]$ with f differentiable on $(-1, 1)$ and $f(-1) = f(1) = 0$ but such that there is no number c in $(-1, 1)$ with $f'(c) = 0$.** [Hint: Which hypothesis above must you violate??]

// A suitable example clearly must fail to be continuous on the interval $[-1, 1]$ and yet satisfy the remaining hypotheses. Here is one such, defined in pieces, of course.

$$f(x) = \begin{cases} 0 & \text{if } x = -1 \text{ or } x = 1 \\ x & \text{if } -1 < x < 1 \end{cases}$$

Then $f(-1) = f(1) = 0$, and $f'(x) = 1$ for every x in the interval $(-1, 1)$. Hence there is no point c in $(-1, 1)$ where $f'(c) = 0$.

3. (10 pts.) Find all the critical points of the function $f(x) = 3 \cdot (x^2 + 8x)^{1/3}$. Which critical points are stationary points? Apply the second derivative test at each stationary point and draw an appropriate conclusion.

First, $f'(x) = (2x + 8)/(x^2 + 8x)^{2/3} = 2(x + 4)/(x(x + 8))^{2/3}$ for $x \neq 0$ and $x \neq -8$. Therefore, f has critical points at $x = -8$, $x = -4$, and $x = 0$. Only $x = -4$ is a stationary point. Observe that we have

$$f''(x) = \frac{[2(x^2 + 8x)^{2/3} - (2x + 8)(2/3)(x^2 + 8x)^{-1/3}(2x + 8)]}{(x^2 + 8x)^{4/3}}$$

for $x \neq 0$ and $x \neq -8$. It follows that $f''(-4) > 0$. Consequently, the second derivative test implies that f has a relative minimum at $x = -4$. [To see $f''(-4) > 0$ easily, note that if $x = -4$, the second term of the numerator of f'' is zero, and what remains is a quotient of squares.]

4. (10 pts.) Locate and determine the maximum and minimum values of the function $f(x) = 2x^3 + 6x^2$ on the interval $[-4, -1]$. What magic theorem allows you to conclude that $f(x)$ has a maximum and minimum even before you attempt to locate them? Why??

Since f is a polynomial, f is continuous on the interval $[-4, -1]$. Consequently, the magical Extreme Value Theorem guarantees that f has absolute extrema of both flavors on $[-4, -1]$. Since we have $f'(x) = 6x^2 + 12x = 6x(x + 2)$, f has only one critical point in $(-4, -1)$, namely $x = -2$. To finish this, it suffices to evaluate f at the critical point and on the boundary of the set $[-4, -1]$ in order to pick out the extrema. Since $f(-1) = 4$, $f(-2) = 8$, and $f(-4) = 2(-4)^3 + 6(-4)^2 = (16)(-2) = -32$, the maximum is 8 and occurs at $x = -2$, and the minimum is -32 and is located at $x = -4$.

5. (10 pts.) (a) State the Mean Value Theorem of Differential Calculus. Use a complete sentence and appropriate notation.

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) , then there is at least one number c in the interval (a,b) with

$$f'(c) = [f(b) - f(a)]/[b - a].$$

(b) Show how to use the Mean Value Theorem to prove the following: If x and y are real numbers, then

$$(*) \quad |\sin(2x) - \sin(2y)| \leq 2|x - y|$$

is true. // First, if $x = y$, then both sides of $(*)$ are zero, and thus the inequality is trivially true. Consequently, it suffices to handle the case where $x \neq y$, where we may assume $x < y$ without any loss of generality. Thus pretend $x < y$. Only a little thought reveals that $f(t) = \sin(2t)$ is continuous on the closed interval $[x,y]$ and differentiable on the interval (x,y) . From the Mean Value Theorem, there must be at least one number c in the interval (x,y) with

$$f'(c) = [f(y) - f(x)]/[y - x].$$

Since $f'(t) = 2 \cdot \cos(2t)$ and $|\cos(2t)| \leq 1$, we have $|\sin(2y) - \sin(2x)|/|y - x| = 2|\cos(2c)| \leq 2$. This, of course, is equivalent to $(*)$.

6. (10 pts.) A small rectangular area of 36 square feet is to be fenced. Three of the sides will use fencing costing \$2.00 per running foot, and the remaining side will use a hedge costing \$1.00 per running foot. Find the dimensions of the rectangle which has the least cost to enclose. Provide a complete enough analysis to convince the doubtful that your extreme value is an absolute minimum. // Denote the lengths of the perpendicular sides of the rectangle by " x " and " y ". For definiteness, let one of the " y " sides and both of the " x " sides cost \$2 per foot, and let the remaining " y " side cost \$1 per foot. [You could, of course, do this the other way around!!] Then $xy = 36$, and the cost in terms of the dimensions is $C = y + 2y + 2x + 2x$ dollars. If we write this in terms of x alone, then we have $C(x) = 3(36/x) + 4x = 4x + 108x^{-1}$ for $x > 0$. [Neither x nor y may be zero. Why?] Now

$$\begin{aligned} C'(x) &= 4 - 108x^{-2} \\ &= (4/x^2) \cdot (x^2 - 27) \\ &= (4/x^2) \cdot (x - (27)^{1/2}) \cdot (x + (27)^{1/2}) \end{aligned}$$

for $x > 0$. Consequently, the only critical point is $x = (27)^{1/2}$. Since we have $C''(x) = 216x^{-3}$, it follows that $C''((27)^{1/2}) > 0$. The Second Derivative Test implies that $C((27)^{1/2})$ is a relative minimum. Since this is the only minimum on $(0,\infty)$ and $C(x)$ is continuous, $C((27)^{1/2})$ is an absolute minimum. The dimensions: $x = (27)^{1/2} = 3(3)^{1/2}$ and $y = (36/(27)^{1/2}) = 12/(3)^{1/2}$ feet.

7. (5 pts.) Find the function $f(x)$ that satisfies the following two equations: $f'(x) = \cos(x) - \sin(x)$ for all x and $f(\pi/2) = 4$.

// Since $f'(x) = \cos(x) - \sin(x)$, it follows that

$$\begin{aligned} f(x) &= \int f'(x) \, dx \\ &= \int \cos(x) - \sin(x) \, dx \\ &= \sin(x) + \cos(x) + c \end{aligned}$$

for some real number c . From what we now know about the structure of f , $f(\pi/2) = 4$ implies that we have $4 = \sin(\pi/2) + \cos(\pi/2) + c$. Solving this little linear equation yields $c = 3$. Thus, $f(x) = \sin(x) + \cos(x) + 3$.

8. (5 pts.) Find a function $g(x)$ so that g satisfies the following equation:

$$\int g(x) dx = \sec^2(x) - 12x + 2e^x + C$$

// From the very definition of antiderivative and how we have agreed to denote such a varmint, it follows that g must satisfy the equation

$$\begin{aligned} g(x) &= (\sec^2(x) - 12x + 2e^x + C)' \\ &= 2 \cdot \sec(x) \cdot \sec(x) \cdot \tan(x) - 12 + 2e^x \end{aligned}$$

9. (10 pts.) Evaluate the following antiderivatives.

$$\int 6x^5 - \frac{7}{x} - \frac{12}{x^2} - \frac{11}{|x|(x^2 - 1)^{1/2}} \, dx = x^6 - 7 \cdot \ln|x| + 12x^{-1} - 11 \cdot \sec^{-1}(x) + C$$

or

$$x^6 - 7 \cdot \ln|x| + 12x^{-1} + 11 \cdot \csc^{-1}(x) + C$$

$$\int e^{2x-3} - \frac{10 \cdot \sin(x)}{\cos^2(x) + 1} \, dx = (1/2)e^{2x-3} + 10 \tan^{-1}(\cos(x)) + C$$

Break this integral into a difference of integrals. The first can be handled with the substitution $u = 2x - 3$ and the second can be done using the substitution $v = \cos(x)$.

10. (20 pts.) Very carefully sketch each of the following functions. Label very carefully.

(a) $f(x)$ is continuous on \mathbb{R} and satisfies the following:

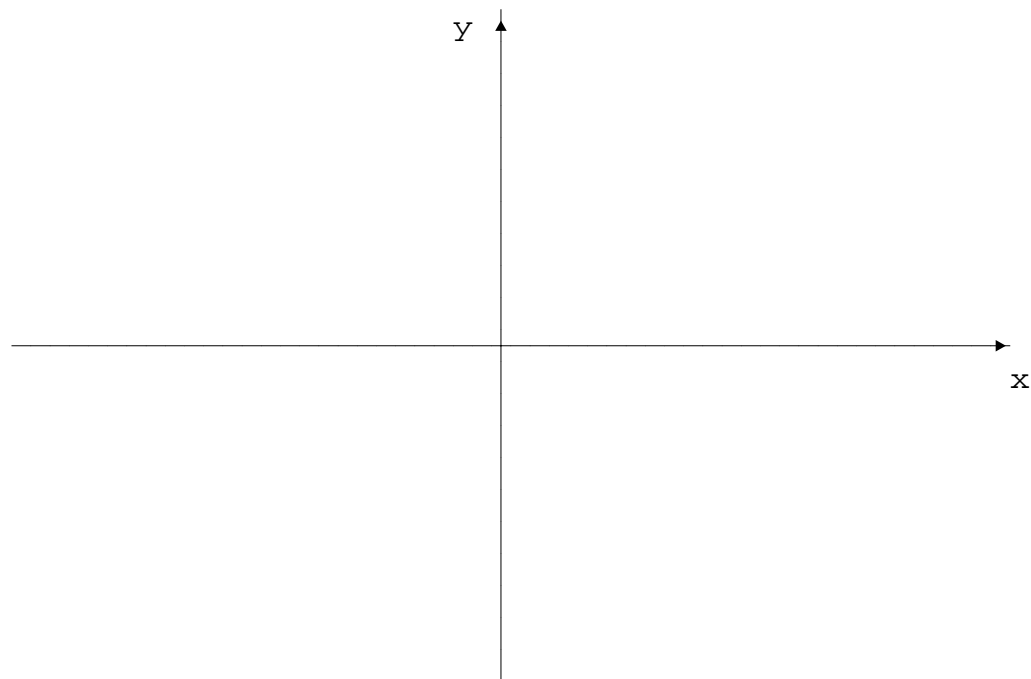
(1) $f(0) = 0$;

(2) $x < 0 \Rightarrow f'(x) > 0$, and $x > 0 \Rightarrow f'(x) < 0$;

(3) $\lim_{x \rightarrow 0^-} f'(x) = +\infty$ and $\lim_{x \rightarrow 0^+} f'(x) = -\infty$;

(4) $x \neq 0 \Rightarrow f''(x) > 0$; and

(5) $f(x) \rightarrow -2$ as $x \rightarrow \pm\infty$. See [c1-t4t-g.htm](#).



(b) $g(x) = 2x/(x^2 + 4)$ [Analyze g' and g'' and how g behaves as $x \rightarrow \pm\infty$. Work on the back of page 4!] See [c1-t4t-g.htm](#).

