Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

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1. (15 pts.) Let $f(x) = 3x(5 - x)$.

(a) Then the slope-predictor function for f at each x is given by $m(x) = 15 - 6x$ since $f(x) = -3x^2 + 15x$.
[Related Exercises: Section 2.1: 1 - 14]

(b) It turns out that the graph of f has a horizontal tangent line for precisely one point on the graph of f, $(x_1,f(x_1))$. What is this ordered pair?

The graph has a horizontal tangent at $(x_1,f(x_1))$ precisely when $m(x_1) = 0$ or $x_1 = 5/2$. Thus $(x_1,f(x_1)) = (5/2, 75/4)$
[Related Exercises: Section 2.1: 15 - 24]

(c) For each real number x_0 that is different from x_1 in part (b) above, the tangent line to the graph of f at $(x_0,f(x_0))$ is not horizontal. As a consequence, it must pass through the x-axis at a unique point $(x_2,0)$. Obtain a formula for x_2 in terms of x_0.

To get started, we first must obtain an equation for the line tangent to the graph of f at $(x_0,f(x_0))$. By using the slope-predictor function in part (a), we can quickly and painlessly produce this: $y - 15x_0 + 3(x_0)^2 = (15 - 6x_0)\cdot(x - x_0)$

Now since $(x_2,0)$ is on the line, after the dust clears we have

$$-15x_0 + 3(x_0)^2 = (15 - 6x_0)\cdot(x_2 - x_0).$$

Thus,
$$x_2 = x_0 + [-15x_0 + 3(x_0)^2]/[(15 - 6x_0)]$$
$$= 3(x_0)^2/[(6x_0 - 15)].$$

[Kindly observe that for this to make sense, it is necessary that x_0 be different from $5/2 = x_1$.]

[Related Exercise: Section 2.1: 28]

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2. (5 pts.) Show how to use the squeeze law of limits to provide an evaluation of the following limit that is completely correct. You will need to show how to build a suitable inequality to provide a complete solution.

$$\lim_{x \to 0} x^2\sin(1/x) = 0$$

To see this, observe that for $x \neq 0$, $-1 \le \sin(1/x) \le 1$ implies that $-x^2 \le x^2\sin(1/x) \le x^2$. Since $x^2 \to 0$ as $x \to 0$, the squeezing theorem implies $x^2\sin(4/x^5) \to 0$ as $x \to 0$. [You need all of this for a complete answer!!]
[Related Exercises: Section 2.3: 25 - 28]

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3. (5 pts.) It turns out that the slope-predictor function for the function $f(x) = \cos(2x)$ is the function $m(x) = -2\cdot\sin(2x)$. Use this to obtain an equation for the line tangent to the graph of $f(x) = \cos(2x)$ at $x_0 = \pi/6$.

$$y - (1/2) = -3^{1/2}\cdot(x - (\pi/6))$$

will do after the dust settles. [Assorted trig treats are needed. For instance, $\sin(\pi/3) = 3^{1/2}/2$ and $\cos(\pi/3) = 1/2$ are useful pieces of trivia.] **[Related Exercises: Section 2.1: 1 - 14 ... building tangent line equations.]**

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4. (10 pts.) (a) Using complete sentences and appropriate notation, state the theorem that is concerned with the intermediate value property of continuous functions.

Suppose that f is continuous on a closed interval [a,b]. If K is any number between f(a) and f(b), then there exists at least one number c in (a,b) such that f(c) = K.

(b) Apply the theorem concerning the intermediate value property of continuous functions to show that the given equation has a solution in the given interval.

$$x^3 + x + 1 = 0 \text{ on } [-1,0]$$

Explain completely. Deal with all the magical hypotheses.

Let $f(x) = x^3 + x + 1$ on [-1,0]. Observe that f(-1) = -1, f(0) = 1, and K = 0 is a number between f(-1) and f(0). Since f is a polynomial, f is continuous on the interval [-1,0]. Consequently, we have satisfied the hypotheses of the theorem that is concerned with the intermediate value property of continuous functions. Thus, we are entitled to invoke the magical conclusion that asserts that there is at least one number c in (-1,0) where f(c) = 0.
[Related Exercises: Section 2.4: 53 - 58]

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5. (5 pts.) Find the value for the constant c, if possible, that will make the function f(x) defined below continuous at $x = \pi$. If you find such a c, using the definition, verify the continuity of f(x) at $x = \pi$. Suppose that

$$f(x) = \begin{cases} c^3 - x^3 & , \quad x \le \pi \\ c\cdot\cos(x/2) & , \quad x > \pi. \end{cases}$$

In order for f to be continuous at $x = \pi$, it is necessary and sufficient for

$$c^3 - \pi^3 = f(\pi) = \lim_{x \to \pi} f(x).$$

Clearly, $$\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} c\cdot\cos(x/2) = 0.$$

Since the two-sided limit at π must have the same value, $c = \pi$.
[Related Exercises: Section 2.4: 49 - 52]

6. (20 pts.) For each of the following, find the limit if the limit exists. If the limit fails to exist, say so. Be as precise as possible here. [Work on the back of page two if you run out of room here.]

(a) $$\lim_{x \to +6} \frac{x + 6}{x^2 - 36} = \lim_{x \to +6} \frac{1}{x - 6} \quad \text{fails to exist.}$$

[Related Exercises: Section 2.3: 29 - 58]

(b) $$\lim_{x \to -6} \frac{x + 6}{x^2 - 36} = \lim_{x \to -6} \frac{1}{x - 6} = -1/12$$

[Related Exercises: Section 2.2: 19 - 28]

(c) $$\lim_{\theta \to 0} \frac{2}{\theta} \cdot \sin\left(\frac{3\theta}{\pi}\right) = \lim_{\theta \to 0} \frac{2 \cdot \sin(3\theta/\pi) \cdot (3/\pi)}{3\theta/\pi} = 6/\pi$$

The key here is to do the algebraic magic to make the expression

$$\frac{\sin(3\theta/\pi)}{3\theta/\pi}$$

appear, since this has the known limit of "1" as $\theta \to 0$.

[Related Exercises: Section 2.3: 1 - 24]

(d) $$\lim_{t \to 0} \frac{(t + 4)^{1/2} - 2}{t} = \lim_{t \to 0} \frac{1}{(t + 4)^{1/2} + 2} = 1/4$$

The key piece of algebraic magic above is captured in the prestidigitation of rationalizing the numerator. Why would one dream of doing that?? Look at the form ... .

[Related Exercises: Section 2.2: 29 - 36]

7. (10 pts.) Suppose that

$$h(x) = \begin{cases} x^2 - 3x & , \text{ if } \quad x < -1 \\ 6 & , \text{ if } \quad x = -1 \\ -4x & , \text{ if } \quad x > -1 \end{cases}$$

Evaluate the following limits:

(a) $\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+} (-4x) = 4$

(b) $\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-} (x^2 - 3x) = 4$

(c) What can you conclude from parts (a) and (b)? [Be as complete as possible. You should be able to discuss at least three concepts that have some bearing here.]

(1) Since the left and right limits are the same, $\lim_{x \to -1} h(x) = 4$

(2) Since the value of the limit is different from h(-1) = 6, the function h fails to be continuous at x = -1.

(3) The discontinuity is removable. Set H(x) = h(x) for $x \neq -1$, and H(-1) = 4. Then clearly H(x) is continuous at x = -1.

[Related Exercises: Section 2.4: 37 - 48]

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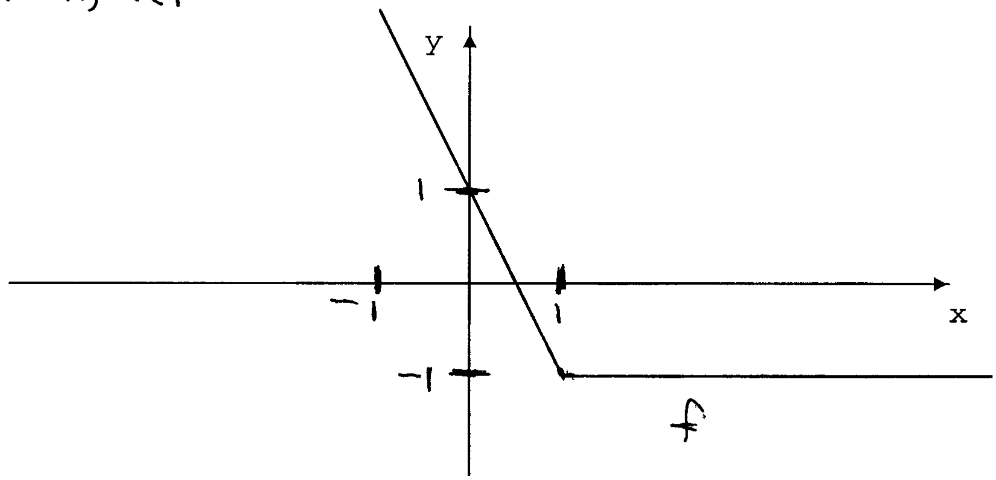
8. (10 pts.) Using an appropriate limit process, show completely how to obtain the slope-predictor function m(x) for the function f(x) = 1/x for $x \neq 0$.

$$\begin{aligned} m(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[1/(x+h)] - [1/x]}{h} \\ &= \lim_{h \to 0} \frac{-h}{h(x+h)x} \\ &= \lim_{h \to 0} -1/[(x + h)x] \\ &= -1/x^2 \end{aligned}$$

[Related Exercises: Section 2.2: 37 - 46]

9. (10 pts.) First write the function $f(x) = |x - 1| - x$ in a piecewise-defined form below. Then sketch its graph below. Label carefully. **[Related Exercises: Section 1.2: 27 - 50]**

$$f(x) = \begin{cases} -1, & x \geq 1 \\ 1-2x, & x < 1 \end{cases}$$



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10. (5 pts.) Using complete sentences and appropriate notation, provide the precise mathematical definitions for each of the following items:

(a) $\lim_{x \to a} f(x) = L$ [Hint: This involves ε and δ.]//

Suppose that f is a function that is defined everywhere in some open interval containing x = a, except possibly at x = a. We write

$$\lim_{x \to a} f(x) = L$$

if L is a number such that for each $\varepsilon > 0$ we can find a $\delta > 0$, such that if x is in the domain of f and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

(b) **Continuity** of a function f(x) at a point x = a // A function f is continuous at x = a if $\lim_{x \to a} f(x) = f(a)$.

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11. (5 pts.) Give an ε - δ proof that $\lim_{x \to 2}(9x - 1) = 17$.

Proof: Let $\varepsilon > 0$ be arbitrary. Set $\delta = \varepsilon/9$. Observe that $\delta > 0$. Suppose now that x satisfies $0 < |x - 2| < \delta$. We shall now verify that $0 < |x - 2| < \delta$ implies $|(9x - 1) - 17| < \varepsilon$. Now

$$\begin{aligned} 0 < |x - 2| < \delta \quad &\Rightarrow \quad |x - 2| < \varepsilon/9 \\ &\Rightarrow \quad 9|x - 2| < \varepsilon \\ &\Rightarrow \quad |9x - 18| < \varepsilon \\ &\Rightarrow \quad |(9x - 1) - 17| < \varepsilon. \end{aligned}$$

Since, given an arbitrary $\varepsilon > 0$, we have produced a number $\delta > 0$ such that, if x satisfies $0 < |x - 2| < \delta$, then $|(9x-1) - 17| < \varepsilon$, we have proved that $(9x - 1) \to 17$ as $x \to 2$. **[Related Exercises: Section 2.3: 75 - 84]**