
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative, $f'(x)$, of a function $f(x)$.// The function f' defined by the equation

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

is called the derivative of f with respect to x. The domain of f' consists of all x in the domain of f for which the limit above exists.

(b) Using the definition of the derivative as a limit, show all steps of the computation of $f'(x)$ when $f(x) = 1/(2x+1)$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[1/(2(x+h)+1)] - [1/(2x+1)]}{h} \\ &= \lim_{h \to 0} \frac{-2h}{h(2(x+h)+1)(2x+1)} \\ &= \lim_{h \to 0} -2/[(2(x + h)+1)(2x+1)] \\ &= -2/(2x+1)^2 \end{aligned}$$

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2. (10 pts.) (a) Using implicit differentiation, compute dy/dx and when $x^5 + y^2 = 9$. **Label your expressions correctly or else.**

$d(x^5 + y^2)/dx = d(9)/dx$ implies that $dy/dx = -5x^4/(2y)$.

(b) Obtain an equation for the line tangent to the graph of $x^5 + y^2 = 9$ at the point $(-1,-(10)^{1/2})$.//Evaluating the implicit derivative above at $(-1,-(10)^{1/2})$ provides us with the slope of the tangent line, namely $5/(2(10)^{1/2})$. Consequently, an equation for the tangent line at the desired point is
$y -(-(10)^{1/2}) = ((10)^{1/2}/4)(x - (-1))$ after a little more algebra.

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3. (10 pts.) (10 pts.) Locate and determine the maximum and minimum values of the function $f(x) = x^3 + 3x^2$ on the interval [-1, 1]. What magic theorem allows you to conclude that f(x) has a maximum and minimum even before you attempt to locate them? Why??//

Since f is a polynomial, f is continuous on the interval [-1,1]. Consequently, the magical Extreme Value Theorem, or what E&P call the Absolute Maxima and Minima Theorem, guarantees that f has absolute extrema of both flavors on [-1,1]. Since we have $f'(x) = 3x^2 + 6x = 3x(x -(- 2))$, f has only one critical point in (-1,1), namely x = 0. To finish this, it suffices to evaluate f at the critical point and on the boundary of the set [-1,1] in order to pick out the extrema. Since f(-1) = 2, f(0) = 0, and f(1) = 4, the maximum is 4 and occurs at x = 1, and the minimum is 0 and is located at x = 0.

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4. (5 pts.) Use logarithmic differentiation to find dy/dx when $y = x^{\tan(x)}$. **Label your expressions correctly or else.**

Since ln(y) = tan(x)ln(x), by doing differentiating both sides and transposing y, we get

$$y' = [\tan(x)\cdot(1/x) + \sec^2(x)\ln(x)]x^{\sec(x)}.$$

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5. (5 pts.) Obtain the following limit. This is easy if you grok the definition of the derivative. [If not, break out your trigonometric tool kit!!]

$$\lim_{h \to 0} \frac{\cos((\pi/3) + h) - \cos(\pi/3)}{h} = (\cos)'(\pi/3)$$

$$= -\sin(\pi/3)$$

$$= -(3)^{1/2}/2$$

6. (20 pts.) Obtain the derivative of each of the following functions. You may use any of the rules of differentiation at your disposal. [4 pts./part]

(a) $f(x) = 4x^5 - 3x^{-6} + 7 \cdot \tan(x)$

$f'(x) = 20x^4 + 18x^{-7} + 7 \cdot \sec^2(x)$

(b) $g(x) = (x^3 - 4x^{-1})\sec(x)$

$g'(x) = (3x^2 + 4x^{-2})\sec(x) + (x^3 - 4x^{-1})\sec(x)\tan(x)$

(c) $h(t) = \dfrac{3\cot(t)}{2t^5 - \csc(t)}$

$$h'(t) = \frac{-3\csc^2(t)(2t^5 - \csc(t)) - (3\cot(t)(10t^4 + \csc(t)\cot(t))}{(2t^5 - \csc(t))^2}$$

(d) $L(x) = \exp(3x^3 - 7x) - 2 \cdot \ln(5x^3 - 14)$

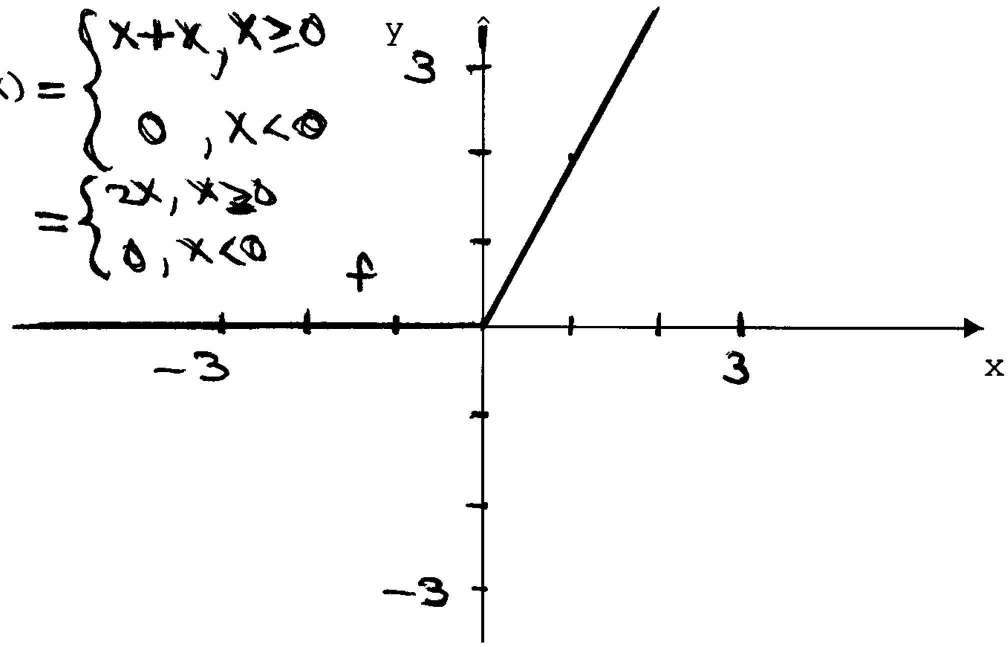
$L'(x) = \exp(3x^3 - 7x)(9x^2 - 7) - 2 \cdot (5x^3 - 14)^{-1}(15x^2)$

(e) $y = 9^x + x^9 + 9^9 + \log_9(x) + \ln(9)$

$$\frac{dy}{dx} = \ln(9)9^x + 9x^8 + 0 + (1/(x \cdot \ln(9))) + 0$$

7. (15 pts.) (a) Sketch the graph of the function $f(x) = x + |x|$. (b) Then investigate its differentiability. Find the derivative where it exists. (c) Find the one-sided derivatives at the points where $f'(x)$ does not exist.

$$f(x) = \begin{cases} x + x, & x \geq 0 \\ 0, & x < 0 \end{cases} = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$$f(x) = \begin{cases} 2x & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

(b) If $x > 0$, then $f(x) = 2x$ implies that $f'(x) = 2$. Likewise, if $x < 0$, then $f(x) = 0$ implies that $f'(x) = 0$. The only possible problem is at $x = 0$ where the two pieces meet. [The graph suggests that this is a place where f fails to be differentiable.]

(c) We shall compute each of the one-sided derivatives of f at $x = 0$.

$$f_+'(0) = \lim_{h \to 0^+} (2(0 + h) - (0))/h = \lim_{h \to 0^+} 2h/h = 2, \text{ and}$$

$$f_-'(0) = \lim_{h \to 0^-} (0 - 0)/h = \lim_{h \to 0^-} 0 = 0.$$

Consequently, $f'(0)$ does not exist.

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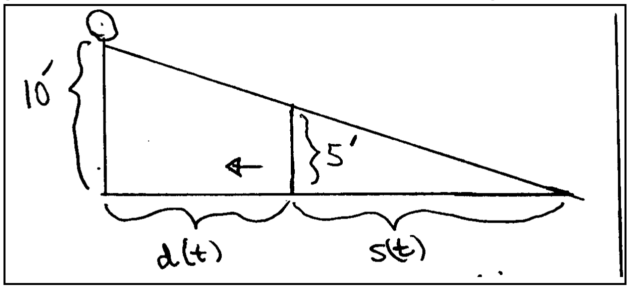
8. (5 pts.) Find all points in the interval $[-\pi,\pi]$ where the graph of $f(x) = x - 2\cdot\sin(x)$ has a horizontal tangent line.
//The graph of f has a horizontal tangent line where $f'(x) = 0$. Now $f'(x) = 1 - 2\cos(x)$. Thus $f'(x) = 0$ in the given interval where $\cos(x) = 1/2$. To deal with this, first find all solutions in the interval $[0,2\pi]$. This turns out to be at two points, $x_0 = \pi/3$, and $x_1 = 5\pi/3$. Thus, all solutions on the real line are given by $x = \pi/3 + 2\pi k$ or $x = 5\pi/3 + 2\pi k$, where k any integer.["Reference Angles" are a help here!] There are two in the interval $[-\pi,\pi]$: $x_0 = \pi/3$, and $x_2 = -\pi/3$.

9. (10 pts.) Suppose a person who is 5 feet tall is walking at night at a speed of 4 feet per second directly toward a street light that is 10 feet high. (a) How fast is the tip of the person's shadow moving along the ground? (b) At what rate is the person's shadow decreasing in length??



Let d(t) denote the distance from the person to the light, in feet, at time t, where t is given in seconds. Using the same units, let s(t) denote the length of the shadow at time t. From the use of similar triangles, it follows that we have s(t)/5 = (s(t) + d(t))/10, from which follows s(t) = d(t).

Observe that d′(t) = -4 feet per second since the distance between the person and the light pole is decreasing.

(a) The rate at which the tip of the shadow is moving along the ground is the rate at which the total distance from the tip of the shadow to the street light is changing. This is s′(t) + d′(t) = 2d′(t) = -8 ft./sec.

(b) s′(t) = d′(t) = -4 feet per second.

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10. (10 pts.) A commuter train carries 600 passengers each day from a suburb to a city. It costs $1.50 per person to ride the train. Market research reveals that 40 fewer people would ride the train for each 5¢ increase in fare, 40 more for each 5¢ decrease in fare. Show completely how to determine which fare should be charged to yield the largest possible revenue, and give the fare that does yield the largest possible revenue. [Warning: Details, details, details...]

// Let r(x) denote the revenue in terms of the fare, x, with both in cents. Then $r(x) = [600 + 40((150 - x)/5)]\cdot x$
$= [600 + 8(150 - x)]\cdot x$.

To ensure $r(x) \geq 0$, it is reasonable to require that $x \geq 0$, and the number of passengers satisfy the same condition, so $600 + 8(150 - x) \geq 0$. Consequently, putting both these inequalities together, a reasonable domain is the interval [0,225]. Since r(x) is a polynomial, it's continuous on this interval. From the theorem that deals with the absolute extrema of continuous functions on closed intervals, we know that we have extrema of both flavors here. A routine bit of algebra reveals that $r'(x) = 1800 - 16x = -16(x - (225/2))$. The only critical point is $x = 225/2$. Since r is non-negative and $r(0) = r(225) = 0$, and $r(225/2) > 0$, we have located the absolute maximum. From the symmetry of the underlying parabola, it is evident that r(112) = r(113). Consequently, either a fare of $1.12 or $1.13 will provide the best real results. [Notes: (a) A messy computation reveals that the value is 101248, in cents. (b) The answer key on line works in terms of the number of 5¢ increases. Consult this source for an alternative answer. Where did the interval [-15,15] arise?? Which solution is to be preferred? For what reasons might you choose one over the other??]