
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (10 pts.) (a) Using implicit differentiation, compute dy/dx and d^2y/dx^2 when $\sin(y) = xy$. **Label your expressions correctly or else.**// This is Problem 21 of Section 4.6. First, we observe that $d(\sin(y))/dx = d(xy)/dx$ implies that $dy/dx = y/(\cos(y) - x)$. Consequently, after differentiating one more time using quotient rule, replacing the occurances of dy/dx in the expression for the second derivative, and cleaning up the algebra, we obtain $d^2y/dx^2 = [2y\cos(y) - 2xy + y^2\sin(y)]/[\cos(y) - x]^3$. Observe that this is equivalent to the answer found on the online E&P goodies.

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2. (5 pts.) Use a linear approximation $L(x)$ to an appropriate function $f(x)$, with an appropriate value of a, to estimate the value of $65^{-2/3}$. [This gem is Problem 29 of Section 4.2.]
// We may use $L(x) = f(a) + f'(a)\cdot(x - a)$ with $f(x) = x^{-2/3}$ and $a = 64 = 2^6$. Performing the requisite prestidigitation results in $65^{-2/3} \approx L(65) = 2^{-4} - (2/3)2^{-10} = (2^5 - (1/3))2^{-9} = 95/1536$.

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3. (5 pts.) Write dy in terms of x and dx when $y = \sin(2x)\cos(3x)$.
$dy = [2\cdot\cos(2x)\cos(3x) - 3\cdot\sin(2x)(\sin(3x)]dx$ after a routine differentiation.

Silly Bonus: Suppose $f''(c) < 0$. By setting $\varepsilon = -f''(c)/2$, using the ε-δ definition of limit on $f''(c)$, and remembering $f'(c) = 0$, we see there must be a number $\delta > 0$ so that if h satisfies $0 < |h| < \delta$, then $|(f'(c + h)/h) - f''(c)| < -f''(c)/2$. Observe that the inequality $|(f'(c + h)/h) - f''(c)| < -f''(c)/2$ is equivalent to $f''(c)/2 < (f'(c + h)/h) - f''(c) < -f''(c)/2$. Thus, by unwrapping the absolute value, we see that if $0 < |h| < \delta$, then $3f''(c)/2 < f'(c + h)/h < f''(c)/2$. By remembering that $f''(c) < 0$, we realize that whenever h is a number that satisfies $0 < |h| < \delta$, $f'(c + h)/h$ must be negative, and so $f'(c + h)$ and h must have different signs.

Let's now see what this tells us about the signs of $f'(x)$ near c. Suppose x is a number that satisfies $c - \delta < x < c$. Then it follows that $-\delta < x - c < 0$. Observe that $x - c$ is a number that satisfies $0 < |x - c| < \delta$ and $x - c < 0$. Thus, from the paragraph above, we have $f'(x) = f'(c + (x - c)) > 0$. Similarly, if x is a number with $c < x < c + \delta$, then $0 < x - c < \delta$. Consequently $0 < |x - c| < \delta$ and $x - c > 0$. Thus, again using oour work from the previous paragraph, we must have $f'(x) = f'(c + (x - c)) < 0$. Here is the picture for purposes of applying the First Derivative Test:

	$f'(x) > 0$		$f'(x) < 0$	
$c-\delta$	↗	c	↘	$c+\delta$

Using the First Derivative Test, f(c) must be a local maximum.

4. (4 pts.) State the Mean Value Theorem of Differential Calculus. Use a complete sentence and appropriate notation.

If f is continuous on the closed interval [a,b] and differentiable on the open interval (a,b), then there is at least one number c in the interval (a,b) with

$$f'(c) = [f(b) - f(a)]/[b - a],$$

or equivalently,

$$f(b) - f(a) = f'(c)\cdot(b - a).$$

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5. (16 pts.) Fill in the blanks of the following analysis with the correct terminology.

Let $f(x) = 4x^3 - x^4$. Then $f'(x) = -4x^3 + 12x^2 = -4x^2(x - 3)$.

Since $f'(x) > 0$ when $0 < x < 3$ or $x < 0$, and f is continuous, f is <u>increasing</u> on the interval $(-\infty, 3)$. Also, because $f'(x) < 0$ for $3 < x$, f is <u>decreasing</u> on the set $(3,\infty)$. Obviously, $x = 0$ and $x = 3$ are <u>critical</u> points of f since $f'(0) = 0$ and $f'(3) = 0$. By using the first derivative test, it follows easily that f has <u>neither a local max nor a local min</u> at $x = 0$, and <u>local maximum [Better: absolute max (Why?)]</u> at $x = 3$.

Since $f''(x) = -12x^2 + 24x = -12x(x - 2)$, we have $f''(0) = 0$, $f''(2) = 0$, $f''(x) > 0$ when $0 < x < 2$, and $f''(x) < 0$ when $x > 2$ or $x < 0$. Thus, f is <u>concave down</u> on the set $(-\infty, 0) \cup (2,\infty)$, f is <u>concave up</u> on the interval $(0,2)$, and f has <u>inflection points</u> at $x = 0$ and $x = 2$.

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9. (a) using L'Hopital's Rule :

(a) $$\lim_{x \to \infty} ((x^2 + 18x)^{1/2} - x) = \lim_{x \to \infty} \frac{(1 + 18x^{-1})^{1/2} - 1}{x^{-1}}$$

$$\overset{(L'H)}{=} \lim_{x \to \infty} \frac{-9x^{-2}(1 + 18x^{-1})^{-1/2}}{-x^{-2}}$$

$$= 9$$ [1st Step: Algebra to 0/0]

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6. (10 pts.) (a) Find all the critical points of the function $f(x) = 3\cdot(x^2 - 2x)^{1/3}$. (b) Apply the second derivative test at each critical point, c, where $f'(c) = 0$, and draw an appropriate conclusion.

First, $f'(x) = (2x - 2)/(x^2 - 2x)^{2/3} = 2(x - 1)/(x(x - 2))^{2/3}$ for $x \neq 0$ and $x \neq 2$. Therefore, f has critical points at $x = 0$, $x = 1$, and $x = 2$. Only at $x = 1$ do we have $f'(x) = 0$. Observe that we have

$$f''(x) = \frac{[2(x^2 - 2x)^{2/3} - (2x - 2)(2/3)(x^2 - 2x)^{-1/3}(2x - 2)]}{(x^2 - 2x)^{4/3}}$$

for $x \neq 0$ and $x \neq 2$. It follows that $f''(1) > 0$. Since $f''(x)$ is continuous in an open interval containing $x = 1$, near $x = 1$ we have $f''(x) > 0$. Consequently, the second derivative test, the weak version found in E&P, implies that f has a relative minimum at $x = 1$. [To see $f''(1) > 0$ easily, note that if $x = 1$, the second term of the numerator of f'' is zero, and what remains is a quotient of squares.]

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7. (5 pts.) Suppose $f(x) = x^{1/2}$ on the interval [100,101]. Show that there is a number c in (100,101) such that

$$101^{1/2} = 10 + (1/2)c^{-1/2}$$

by using the Mean Value Theorem. [Take care to tell me about the hypotheses that must be satisfied before you assert the conclusion is true for the function at hand.] // Observe that f is continuous on [100,101] and differentiable on (100,101) with $f'(x) = (1/2)x^{-1/2}$. From the Mean Value Theorem, there is a number c in (100,101) with $f(101)-f(100)=f'(c)(101-100)$, or after substituting and doing the arithmetic, $101^{1/2} - 10 = (1/2)c^{-1/2}$, which is equivalent to the desired equation.

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8. (5 pts.) Find the function f(x) that satisfies the following two equations: $f'(x) = 2\cdot\sec^2(x) + (4/\pi)$ for every real number x in $(-\pi/2,\pi/2)$, and $f(\pi/4) = 8$.

// Since $f'(x) = 2\cdot\sec^2(x) + (4/\pi)$, it follows that

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int 2\cdot\sec^2(x) + (4/\pi)\,dx \\ &= 2\cdot\tan(x) + (4/\pi)x + c \end{aligned}$$

for some real number c. From what we now know about the structure of f, $f(\pi/4) = 8$ implies that

$$8 = 2\cdot\tan(\pi/4) + (4/\pi)\cdot(\pi/4) + c$$

Solving this little linear equation yields $c = 5$. Thus,

$$f(x) = 2\cdot\tan(x) + (4/\pi)x + 5.$$

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9. (10 pts.) Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible.

(a) $\lim_{x \to \infty} ((x^2 + 18x)^{1/2} - x) = \lim_{x \to \infty} \dfrac{18x}{(x^2 + 18x)^{1/2} + x}$

$$= \lim_{x \to \infty} \frac{18x}{|x|(1 + 18x^{-1})^{1/2} + x}$$

$$= \lim_{x \to \infty} \frac{18}{(1 + 18x^{-1})^{1/2} + 1}$$

$= 18/2 = 9$ [Alternative: Page 2.]

(b) $\lim_{x \to 0} \dfrac{\exp(x^3) - 1}{x - \sin(x)} = \lim_{x \to 0} \dfrac{3x^2\exp(x^3)}{1 - \cos(x)}$

$$= \lim_{x \to 0} \frac{6x \cdot \exp(x^3) + 9x^4 \cdot \exp(x^3)}{\sin(x)}$$

$= 6$

after you use l'Hopital's Rule twice and remember $x/\sin(x) \longrightarrow 1$ as $x \longrightarrow 0$ **or** simply use l'Hopital's Rule three times.

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10. (10 pts.) Evaluate each of the following anti-derivatives.

(a)

$$\int 7x^6 - \frac{\pi}{x} - \frac{12}{x^3} + 5 \cdot \cos(10x)\, dx =$$

$$x^7 - \pi \cdot \ln|x| + 6x^{-2} + (1/2)\sin(10x) + C$$

(b)

$$\int e^{2x} + e^{-2x} - \frac{2x^4 - 3x^3 + 5}{7x^2}\, dx =$$

$$(1/2)e^{2x} - (1/2)e^{-2x} - (2/21)x^3 + (3/14)x^2 - (5/7)x^{-1} + C$$

after doing a little algebraic magic on the third term of the integrand --- divide and conquer.

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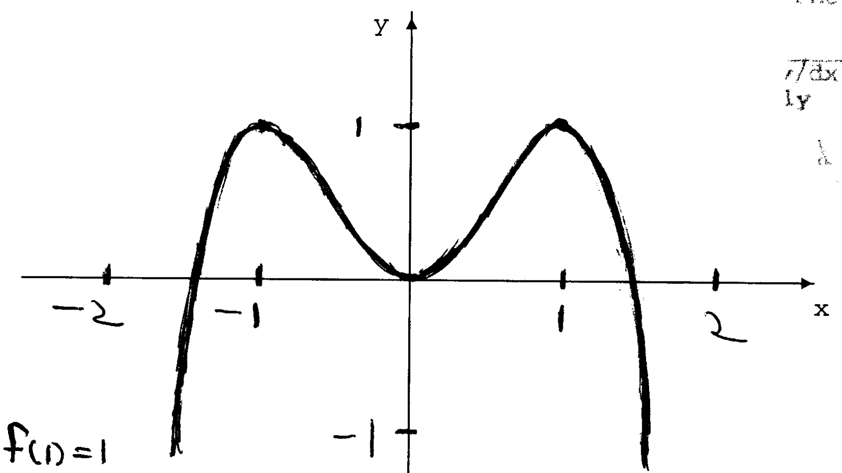
11. (10 pts.) Sketch the graph of $f(x) = 2x^2 - x^4$. At the very least you should determine the critical points of f and intervals where f is increasing or decreasing to do this. [If you run out of room here in doing your analysis, work on the back of Page 4.]



$\begin{cases} f(-1) = f(1) = 1 \\ f(0) = 0 \end{cases}$

$f(x) = 0 \iff x = \pm\sqrt{2}$ or $x = 0$

$f'(x) = 4x - 4x^3$
$= 4x(1-x^2) = -4x(x-1)(x+1)$

$f'>0$ | -1 | $f'<0$ | 0 | $f'>0$ | 1 | $f'<0$

$\nearrow$ -1 $\searrow$ 0 $\nearrow$ 1 $\searrow$

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12. (10 pts.) Determine the open intervals where the function $f(x) = x^3(x - 1)^4$ is concave up and concave down. Then locate any inflection points. [Note: Do not attempt to graph this varmint. Communicate your results using complete sentences.] Note: This is part of Problem 69 of Section 4.6. // After differentiating twice very carefully and using the quadratic formula to complete the factorization of the second derivative, you should be looking at something resembling this:

$$f''(x) = 42x(x-1)^2(x - ((3 - 2^{1/2})/7)) \cdot (x - ((3 + 2^{1/2})/7))$$

The roots of f'' in order are 0, $(3 - 2^{1/2})/7$, $(3 + 2^{1/2})/7$, and 1. By analysing the sign of second derivative, you can see that f is concave down on the set $(-\infty , 0) \cup ((3 - 2^{1/2})/7, (3 + 2^{1/2})/7)$ and concave up on $(0 , (3 - 2^{1/2})/7) \cup ((3 + 2^{1/2})/7, \infty)$. Consequently f has inflection points at 0, $(3 - 2^{1/2})/7$, and $(3 + 2^{1/2})/7$, but not at $x = 1$.

Silly Ten Point Bonus: Pretend that f is differentiable in an open interval I that contains the number c with $f'(c) = 0$. Suppose that $f''(c) < 0$. By unraveling the definition of $f''(c)$ in terms of epsilons and deltas, show that there is a $\delta > 0$ such that $f'(c + h)$ and h have different signs if $0 < |h| < \delta$. What does this imply about the signs of $f'(x)$ near c. [Details on Page 1 ...]