Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

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1. (10 pts.) Using the text's four step process, show completely how to obtain the slope-predictor function m(x) for the function $f(x) = 1/x^2$ for $x \neq 0$.

$$m(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[1/(x+h)^2] - [1/x^2]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 - [x^2+2xh+h^2]}{h(x+h)^2x^2}$$

$$= \lim_{h \to 0} -(2x + h)/[(x + h)^2x^2]$$

$$= -2/x^3$$

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2. (10 pts.) Let $f(x) = 2x(x + 3)$.

(a) Then the slope-predictor function for f is given by

$m(x) = 4x + 6$ since $f(x) = 2x^2 + 6x$.

(b) It turns out that the graph of f has a horizontal tangent line at precisely one point on the graph of f, $(x_1,f(x_1))$. What is this ordered pair?

Plainly f has a horizontal tangent line at $(x_1,f(x_1))$ precisely when the slope-predictor function of f at x_1 is zero. Now

$$m(x_1) = 0 \Leftrightarrow 4x_1 + 6 = 0 \Leftrightarrow x_1 = -3/2.$$

Thus, computing the value of $f(x_1)$, we have

$$(x_1,f(x_1)) = (-3/2, -9/2)$$

3. (5 pts.) It turns out that the slope-predictor function for the function $f(x) = \tan(x)$ is the function $m(x) = \sec^2(x)$. Use this to obtain an equation for the line tangent to the graph of $f(x) = \tan(x)$ at $x_0 = \pi/4$.

Since $f(\pi/4) = \tan(\pi/4) = 1$ and $m(\pi/4) = \sec^2(\pi/4) = 2$, an equation for the line tangent to the graph of f at $x_0 = \pi/4$ is

$$y - 1 = 2(x - \pi/4).$$

There are, of course, infinitely many equations equivalent to this one.

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4. (15 pts.) (a) First write the function

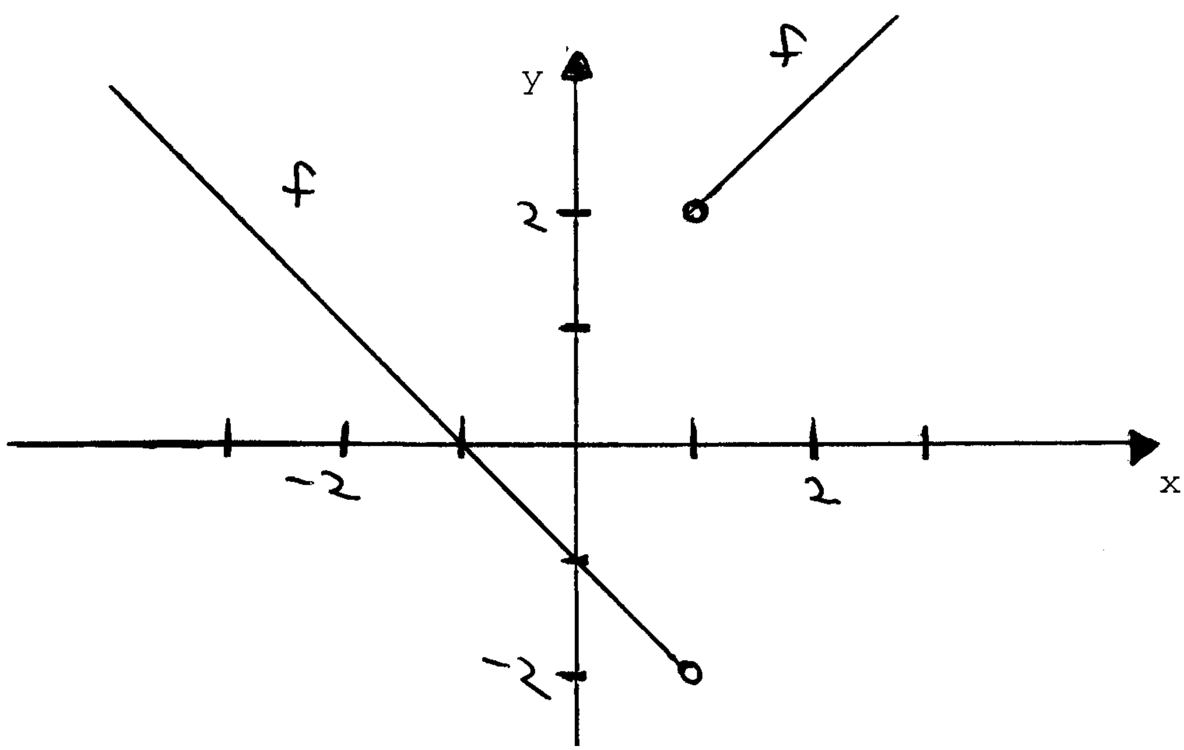
$$f(x) = \frac{x^2 - 1}{|x - 1|}$$

in a piecewise-defined form below. (b) Then sketch its graph. Label carefully. (c) Evaluate each one-sided limit at x = 1. (d) Using part (c), what can you say about the existence of the two-sided limit at x = 1??

(a)

$$f(x) = \begin{cases} (x^2 - 1)/(x-1), & x > 1 \\ (x^2 - 1)/(1-x), & x < 1 \end{cases} = \begin{cases} x+1, & x > 1 \\ -(x+1), & x < 1. \end{cases}$$

(b)



(c)

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x+1) = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -(x+1) = -2$$

(d) Since the two one-sided limits have different values, the two-sided limit does not exist.

5. (20 pts.) For each of the following, find the limit if the limit exists. If the limit fails to exist, say so. Be as precise as possible here. [Work on the back of page two if you run out of room here.]

(a) $$\lim_{x \to -9} \frac{x + 9}{x^2 - 81} = \lim_{x \to -9} \frac{1}{x - 9} = -1/18$$

(b) $$\lim_{x \to +9} \frac{x + 9}{x^2 - 81} = \lim_{x \to +9} \frac{1}{x - 9} \quad \text{fails to exist.}$$

(c) $$\lim_{\theta \to 0} \frac{\tan(5\pi\theta)}{\sin(2\theta)} = \lim_{\theta \to 0} \frac{\sin(5\pi\theta) \times 5\pi \times 2\theta}{5\pi\theta \cdot \cos(5\pi\theta)\sin(2\theta) \cdot 2} = 5\pi/2$$

(d) $$\lim_{x \to 4} \frac{x - 4}{2 - x^{1/2}} = \lim_{x \to 4} \frac{(x^{1/2}+2)(x^{1/2}-2)}{2 - x^{1/2}} = -\lim_{x \to 4}(x^{1/2}+2) = -4$$

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6. (10 pts.) Where does the straight line tangent to the graph of $y = x^2$ at the point (-10,100) intersect the x-axis??

Since the slope-predictor function for $y = x^2$ is $m(x) = 2x$ for each x, an equation for the tangent line at (-10,100) is given by $y - 100 = m(-10)(x - (-10))$ or $y - 100 = -20(x + 10)$. An equivalent equation in slope-intersect form is $y = -20x - 100$. This line intersects the x-axis when $y = 0$. Solving for the x-component yields $x = -5$. Thus, the point of intersection is (-5,0).

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7. (5 pts.) Using complete sentences and appropriate notation, provide the precise mathematical definitions for each of the following items:

$\lim_{x \to a} f(x) = L$ [Hint: This involves ε and δ.]//

Suppose that f is a function that is defined everywhere in some open interval containing x = a, except possibly at x = a. We write

$$\lim_{x \to a} f(x) = L$$

if L is a number such that for each $\varepsilon > 0$, we can find a $\delta > 0$, such that if x is in the domain of f and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

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8. (5 pts.) Give a complete ε - δ proof that

$$\lim_{x \to -3}(7x - 9) = -30.$$

Proof: Let $\varepsilon > 0$ be arbitrary. Set $\delta = \varepsilon/7$. Observe that $\delta > 0$. Suppose now that x satisfies $0 < |x - (-3)| < \delta$. We shall now verify that $0 < |x - (-3)| < \delta$ implies $|(7x - 9) - (-30)| < \varepsilon$. Now

$$\begin{aligned} 0 < |x - (-3)| < \delta &\Rightarrow |x + 3| < \varepsilon/7 \\ &\Rightarrow 7|x + 3| < \varepsilon \\ &\Rightarrow |7x + 21| < \varepsilon \\ &\Rightarrow |(7x - 9) - (-30)| < \varepsilon. \end{aligned}$$

Since, given an arbitrary $\varepsilon > 0$, we have produced a number $\delta > 0$ such that, if x satisfies $0 < |x - (-3)| < \delta$, then $|(7x-9) - (-30)| < \varepsilon$, we have proved $(7x-9) \to -30$ as $x \to -3$.

9. (5 pts.) Show how to use the squeeze law of limits to provide an evaluation of the following limit that is completely correct. You will need to show how to build a suitable inequality to provide a complete solution.

$$\lim_{x \to 0} x^2\cos^2(1/x) = 0$$

To see this, observe that for $x \neq 0$, $-1 \leq \cos(1/x) \leq 1$ implies that $0 \leq x^2\cos^2(1/x) \leq x^2$ because $\cos^2(1/x) \geq 0$. Since $x^2 \to 0$ as $x \to 0$, the squeezing theorem implies $x^2\cos^2(1/x) \to 0$ as $x \to 0$. [You need all of this, or something equivalent for a complete answer!!]

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10. (5 pts.) Suppose that

$$h(x) = \begin{cases} x^2 - 3x & \text{, if } \quad x < 1 \\ 6 & \text{, if } \quad x = 1 \\ -2x & \text{, if } \quad x > 1 \end{cases}$$

Evaluate the following limits:

(a) $\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} (-2x) = -2$

(b) $\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} (x^2 - 3x) = -2$

(c) What can you conclude from parts (a) and (b)? Why??

Since the two one-sided limits at $x = 1$ are the same, -2, we actually have

$$\lim_{x \to 1} h(x) = -2.$$

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11. (5 pts.) Show a complete evaluation of the following somewhat thorny limit:

$$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta \cdot \sin(2\theta)} = \lim_{\theta \to 0} \frac{\sin^2(\theta)}{2\theta \cdot \sin(\theta)\cos(\theta)(1 + \cos(\theta))}$$

$$= \lim_{\theta \to 0} \frac{\sin(\theta)}{2\theta \cdot \cos(\theta)(1 + \cos(\theta))} = 1/4$$

This uses both the obvious Pythagorean identity for sine and cosine and $\sin(2x) = 2\sin(x)\cos(x)$. There are other, more devious ways to deal with this using $\sin^2(x/2) = [1 - \cos(x)]/2$.

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12. (5 pts.) What is the slope of the line normal to the curve $y = 2x^2 + 3x - 5$ at the point $P(2,9)$??

Since the slope-predictor function is $m(x) = 4x + 3$, the slope of the line tangent to the graph at $(2,9)$ is $m(2) = 11$. Thus, the slope of the normal line is $m = -1/11$.