
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative, $f'(x)$, of a function $f(x)$.// The function f' defined by the equation

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

is called the derivative of f with respect to x. The domain of f' consists of all x in the domain of f for which the limit above exists.

(b) Using the definition of the derivative as a limit, show all steps of the computation of $f'(x)$ when $f(x) = 1/(3 - x)$.
FIX

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[1/(3 - (x+h))] - [1/(3 - x)]}{h} \\ &= \lim_{h \to 0} \frac{h}{h(3-x-h)(3-x)} \\ &= \lim_{h \to 0} 1/[(3-x-h)(3-x)] \\ &= 1/(3 - x)^2 \end{aligned}$$

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2. (5 pts.) The height of a ball thrown vertically upward is given by $y(t) = -16t^2 + 160t$ for $0 \le t \le 10$. Find the maximum height the ball attains.

The function $y(t)$ is continuous on $[0,10]$ and therefore must have extrema of both types on $[0,10]$. Observe that we have $y'(t) = -32t + 160$, and that $y'(t) = 0$ if, and only if $t = 5$. Since $y(0) = y(10) = 0$ and $y(5) = -16(5)^2 + 160(5) = (5)(160 - 80) = (5)(80) = 400$, the maximum height is 400.

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3. (5 pts.) Using a complete sentence and appropriate notation, provide the precise mathematical definition for the following:

Continuity of a function $f(x)$ at a point $x = a$ // A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

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4. (10 pts.) Locate and determine the maximum and minimum values of the function $f(x) = x^3 + 3x^2$ on the interval [-1, 1]. What magic theorem allows you to conclude that f(x) has a maximum and minimum even before you attempt to locate them? Why??

Since f is a polynomial, f is continuous on the interval [-1,1]. Consequently, the magical Extreme Value Theorem, or what E&P call the Absolute Maxima and Minima Theorem, guarantees that f has absolute extrema of both flavors on [-1,1]. Since we have $f'(x) = 3x^2 + 6x = 3x(x -(- 2))$, f has only one critical point in (-1,1), namely x = 0. To finish this, it suffices to evaluate f at the critical point and on the boundary of the set [-1,1] in order to pick out the extrema. Since $f(-1) = 2$, $f(0) = 0$, and $f(1) = 4$, the maximum is 4 and occurs at x = 1, and the minimum is 0 and is located at x = 0.

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5. (5 pts.) What can you say about the number of critical points of the general cubic function $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$? Why?? [Details, details, detail...]

The general cubic function is differentiable on the whole real line with $f'(x) = 3ax^2 + 2bx + c$. Consequently the number of critical points of f is the number of zeros of the quadratic function $f'(x) = 3ax^2 + 2bx + c$. From elementary algebra, we know that the nature of the zeros of a quadratic is determined by the sign of discriminant of the polynomial. The discriminant of the quadratic polynomial $3ax^2 + 2bx + c$ is $D = (2b)^2 - 4(3a)(c) = 4(b^2 - 3ac)$. Therefore f has two distinct real critical points if $D > 0$, one real critical point if $D = 0$, and no real critical points if $D < 0$.

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6. (5 pts.) Give an example of a continuous function f(x) which has a point c in its domain with $f'(c) = 0$, but such that f(c) is not a local extreme value.

Perhaps the simplest example of such a function is $f(x) = x^3$ which has a single critical point at c = 0 with f(0) = 0 neither a local maximum nor a local minimum. That 0 is neither is clear since $x < 0$ implies that $x^3 < 0$, and $0 < x$ implies that $0 < x^3$.

6. (20 pts.) Obtain the derivative of each of the following functions. You may use any of the rules of differentiation at your disposal. Do not simplify the algebra. [4 pts./part]

(a) $f(x) = 7x^5 - 3x^{-5} + 20 \cdot x^{1/5}$

$f'(x) = 35x^4 + 15x^{-6} + 4x^{-4/5}$

(b) $g(x) = (x^3 - 4x^{-1})(256x^2 - 6x^{-7})$

$g'(x) = (3x^2 + 4x^{-2})(256x^2 - 6x^{-7}) + (x^3 - 4x^{-1})(512x + 42x^{-8})$

(c) $h(t) = \dfrac{3t^2 + 7}{2t^5 - t^{-5}}$

$$h'(t) = \frac{(6t)(2t^5 - t^{-5}) - (3t^2 + 7)(10t^4 + 5t^{-6})}{(2t^5 - t^{-5})^2}$$

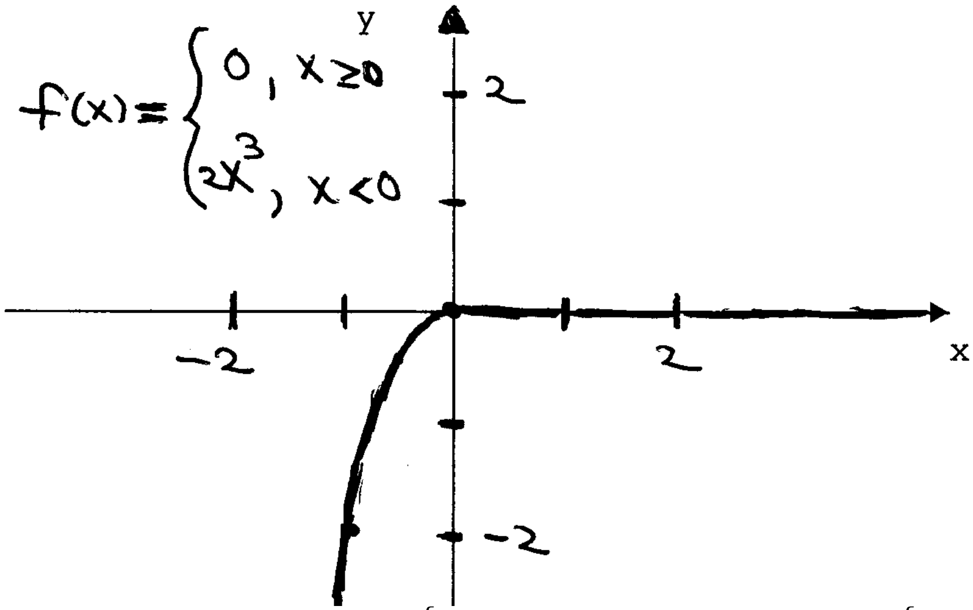
(d) $L(x) = (x^4 + 3x^{-2})^{7/2}$

$L'(x) = (7/2)(x^4 + 3x^{-2})^{5/2}(4x^3 - 6x^{-3})$

(e) $y = (x^7 - 2x^{-2})^{24}(x^{3/2} + 10x)^5$

$$\frac{dy}{dx} = 24(x^7 - 2x^{-2})^{23}(7x^6 + 4x^{-3})(x^{3/2} + 10x)^5 + (x^7 - 2x^{-2})^{24}5(x^{3/2} + 10x)^4((3/2)x^{1/2} + 10)$$

7. (15 pts.) (a) Sketch the graph of the function $f(x) = x^3 - |x|^3$. (b) Find the derivative for $x > 0$ and $x < 0$. (c) Show the computation of the one-sided derivatives at $x = 0$. What can you conclude from this??



(a):

$$f(x) = x^3 - |x|^3 = \begin{cases} x^3-(x)^3 & , x \geq 0 \\ x^3-(-x)^3 & , x < 0 \end{cases} = \begin{cases} 0 & , x \geq 0 \\ 2x^3 & , x < 0. \end{cases}$$

(b):

$$f'(x) = \begin{cases} 0 & , x > 0 \\ 6x^2 & , x < 0. \end{cases}$$

(c):

$$f_-'(0) = \lim_{h \to 0^-} [f(0+h) - f(0)]/h = \lim_{h \to 0^-} 2(h)^3/h = 0$$

$$f_+'(0) = \lim_{h \to 0^+} [f(0+h) - f(0)]/h = \lim_{h \to 0^+} 0/h = 0$$

Thus, $f'(0) = 0$.

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8. (5 pts.) Find the all values for the constant c, if possible, that will make the function f(x) defined below continuous at $x = 0$.

$$f(x) = \begin{cases} \tan(9x)/x & , \quad x > 0 \\ c^2 \cdot \cos(x/2) & , \quad x \leq 0. \end{cases}$$

In order for f to be continuous at $x = 0$, it is necessary and sufficient for $c^2 = f(0) = \lim_{x \to 0} f(x)$.

Clearly, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \tan(9x)/x = 9 \cdot \lim_{x \to 0^+} \tan(9x)/9x = 9.$

Since the two-sided limit at 0 must have the same value, $c^2 = 9$. Hence, $c = 3$ or $c = -3$ will do the job. Observe that the left-handed limit is not a problem:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} c^2 \cdot \cos(x/2) = c^2.$$

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9. (10 pts.) (a) Using complete sentences and appropriate notation, state the theorem that is concerned with the intermediate value property of continuous functions.

Suppose that f is continuous on a closed interval [a,b]. If K is any number between f(a) and f(b), then there exists at least one number c in (a,b) such that f(c) = K.

(b) Apply the theorem concerning the intermediate value property of continuous functions to show that the given equation has a solution in the given interval.

$$x^5 - 5x^3 + 3 = 0 \text{ on } [-1,1]$$

Explain completely. Deal with all the magical hypotheses.

Let $f(x) = x^5 - 5x^3 + 3$ on [-1,1]. Observe that f(-1) = 7, f(1) = -1, and K = 0 is a number between f(-1) and f(1). Since f is a polynomial, f is continuous on the interval [-1,1]. Consequently, the function f satisfies the hypotheses of the theorem that is concerned with the intermediate value property of continuous functions on the interval [-1,1]. Thus, we are entitled to invoke the magical conclusion that asserts that there is at least one number c in (-1,1) where f(c) = 0.

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10. (10 pts.) Find all points on the graph of

$$f(x) = x^{4/3} - 8x^{1/3}$$

where the tangent line is either horizontal or vertical, and clearly say which points have horizontal tangent lines and which have vertical tangent lines.

$f'(x) = (4/3)x^{1/3} - (8/3)x^{-2/3}$ for $x \neq 0$. By performing a little ordinary algebraic magic, we may re-write f′ in the more revealing form $f'(x) = [4(x - 2)]/[3x^{2/3}]$. Since f′(x) = 0 if, and only if x = 2, f has a horizontal tangent line at $(2,f(2)) = (2, -6(2)^{1/3})$ on the graph of f. Evidently, since zero is in the domain of f but not in the domain of f′, we should examine the limit behaviour of $|f'(x)|$ as $x \to 0$. Since

$$|f'(x)| = |[4(x - 2)]/[3x^{2/3}]| \to \infty \text{ as } x \to 0,$$

(0, f(0)) = (0,0) is a point on the graph of f with a vertical tangent line.