
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (10 pts.) (a) Using implicit differentiation, compute dy/dx and when $x^3 + y^2 = 9$. **Label your expressions correctly or else.**

First, we observe that $d(x^3 + y^2)/dx = d(9)/dx$ implies that $3x^2 + 2y(dy/dx) = 0$. Thus, $dy/dx = -3x^2/2y$.

(b) Obtain an equation for the line tangent to the graph of $x^3 + y^2 = 9$ at the point $(-1,-(10)^{1/2})$.

Since

$$\left.\frac{dy}{dx}\right|_{(-1,-\sqrt{10})} = \frac{3}{2\sqrt{10}} \ ,$$

an equation for the line tangent to the graph at $(-1,-(10)^{1/2})$ is

$$y - (-\sqrt{10}) = \frac{3}{2\sqrt{10}}(x - (-1)) \ .$$

Of course there are a multitude of equations that are equivalent to this one.

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2. (5 pts.) Use a linear approximation L(x) to an appropriate function f(x), with an appropriate value of a, to estimate the value of ln(11/10).

We may use $L(x) = f(a) + f'(a)\cdot(x - a)$ with $f(x) = \ln(x)$ and $a = 1$. Performing the requisite prestidigitation results in $\ln(11/10) \approx L(11/10) = 1/10$ since $f(1) = 0$ and $f'(1) = 1$.

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3. (5 pts.) Write dy in terms of x and dx when $y = \sin(2x)\cos(3x)$.

$dy = [2\cdot\cos(2x)\cdot\cos(3x) - 3\cdot\sin(2x)\cdot\sin(3x)]dx$

4. (5 pts.) If you want to find the point (x_0,y_0) on the line defined by $2x + y = 3$ that is closest to the point (3,2), what function should you minimize on what domain?? [Note: Simply obtain the function and its domain. DON'T attempt to analyze the function.]

There are a couple of natural candidates for a function. The first gives the distance from a point (x,y) on the line to the point (3,2), namely

$$f(x) = \left((x - 3)^2 + ((3 - 2x) - 2)^2 \right)^{1/2},$$

defined on $(-\infty,\infty)$. A variant that is slightly more user-friendly is the square of the distance from a point (x,y) on the line to the point (3,2), namely

$$g(x) = (x - 3)^2 + ((3 - 2x) - 2)^2 ,$$

also defined on $(-\infty,\infty)$.

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5. (5 pts.) Use logarithmic differentiation to find dy/dx when $y = x^{\ln(x)}$. **Label your expressions correctly or else.**

Since $\ln(y) = \ln(x)\cdot\ln(x) = \ln^2(x)$, $(1/y)(dy/dx) = 2\cdot\ln(x)(1/x)$. Thus,

$$dy/dx = (2/x)\ln(x)x^{\ln(x)}.$$

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6. (5 pts.) Find the function f(x) that satisfies the following two equations: $f'(x) = x^{-2}$ and $f(1) = 1$.

First notice that if $h(x) = x^{-1}$, then $h'(x) = -x^{-2}$. Thus, if we let $g(x) = -h(x) = -x^{-1}$, then $g'(x) = x^{-2}$. This means that g and f have the same derivatives on $(0,\infty)$. It follows from the Mean Value Theorem that $f(x) - g(x) = K$, for some constant K. [The reason for this is that the function f - g has a zero derivative on the interval and so must be a constant function.] All we need do is to determine K in order to completely identify f. Since $f(1) = 1$ and $g(1) = -1$, $K = 2$. Thus, $f(x) = g(x) + 2 = 2 - x^{-1}$.

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7. (5 pts.) Compute the differential of each side of the given equation, regarding x and y as dependent variables (as if both were functions of some third, unspecified, variable). Then solve for dy/dx.

$$x^3 + y^3 = 3xy$$

Think about t as the independent variable in question, but don't suffer from a case of the dt's. What you get by differentiating with respect to t but not writing dt below dx and dy is

$$3x^2dx + 3y^2dy = 3\cdot dx\cdot y + 3\cdot x\cdot dy.$$

By dividing by $3\cdot dx$ and solving for dy/dx, you should obtain something resembling

$$dy/dx = [y - x^2]/[y^2 - x].$$

8. (10 pts.) Differentiate each of the following functions. Do not attempt to simplify the algebra. **Label your derivatives correctly or else.**

(a) $g(t) = [\ln(t) + e^{-t}]/\sin(3t)$

$g'(t) = [(t^{-1} - e^{-t})(\sin(3t)) - 3(\cos(3t))(\ln(t) + e^{-t})]/\sin^2(3t)$

(b) $f(t) = t\cdot\sec(t)\cdot\tan(t)$

$f'(t) = \sec(t)\tan(t) + t\cdot(\sec(t)\tan(t))\tan(t) + t\cdot\sec(t)\sec^2(t)$

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9. (5 pts.) Find all points in the interval $[0,2\pi]$ where the graph of $f(x) = x - 2\cdot\cos(x)$ has a horizontal tangent line.

$f'(x) = 1 + 2\cdot\sin(x)$ and the graph of f has a horizontal tangent line precisely when $f'(x) = 0$ in $[0,2\pi]$. Now $f'(x) = 0$ if, and only if $\sin(x) = -1/2$ in $[0,2\pi]$. Evidently the reference angle is $\pi/6$. Since sine is negative in the third and fourth quadrants, the desired numbers are $x_0 = \pi + (\pi/6) = 7\pi/6$ and $x_1 = 2\pi - (\pi/6) = 11\pi/6$.

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10. (5 pts.)

Suppose $f(x) = x^{1/2}$ on the interval [81,82]. Show that there is a number c in (81,82) such that

$$82^{1/2} = 9 + (1/2)c^{-1/2}$$

by using the Mean Value Theorem. [Take care to tell me about the hypotheses that must be satisfied before you assert the conclusion is true for the function at hand. How the incantation is muttered is significant.]

Plainly, f is continuous on the closed interval [81,82] and differentiable on the interval (81,82) with $f'(x) = (1/2)x^{-1/2}$. Consequently, f satisfies the hypotheses of the Mean Value Theorem on [81,82]. It follows that there is at least one number c in the interval (81,82) with

$$f'(c) = [f(82) - f(81)]/[82 - 81],$$

or equivalently,

$$82^{1/2} - 9 = (1/2)(c)^{-1/2}.$$

This last equation evidently is equivalent to the one we want.

11. (10 pts.) The height of a cone is decreasing at 3 cm/sec. while its radius is increasing at 2 cm/sec. When the radius is 8 cm and the height is 12 cm, what is the rate at which the volume is changing? Is the volume increasing or decreasing then?? Which??? [Hint: $V = (1/3)\pi r^2 h$]

Let t_0 denote the time when the radius is 8 cm. and the height is 12 cm. Since $V(t) = (\pi/3)r^2(t)h(t)$,

$$V'(t) = (2\pi/3)r(t)r'(t)h(t) + (\pi/3)r^2(t)h'(t).$$

Now $h'(t) = -3$ and $r'(t) = 2$. Thus, the rate at which the volume is changing at time t_0 is given by

$$\begin{aligned} V'(t_0) &= (2\pi/3)r(t_0)r'(t_0)h(t_0) + (\pi/3)r^2(t_0)h'(t_0). \\ &= (2\pi/3)(8)(2)(12) + (\pi/3)(64)(-3) \\ &= 64\pi \text{ cm/sec.} \end{aligned}$$

Since the instantaneous rate of change is positive, we'd expect the volume to be increasing.

12. (10 pts.) Find and classify the critical points of the function $f(x) = \sin^3(x)$ in the interval $(-3,3)$.

Since $f'(x) = 3\cdot\sin^2(x)\cdot\cos(x)$, the critical points of f in the interval $(-3,3)$, are $x = -\pi/2$, $x = 0$, and $x = \pi/2$. Because $\sin^2(x)$ is positive when x is nonzero in $(-3,3)$, the sign of the derivative is controlled by $\cos(x)$. It follows that $f'(x) < 0$ when $-3 < x < -\pi/2$ or $\pi/2 < x < 3$, and that $f'(x) > 0$ when we have $-\pi/2 < x < 0$ or $0 < x < \pi/2$. It follows from the first derivative test that $f(-\pi/2) = -1$ is a global minimum, $f(\pi/2) = 1$ is a global maximum, but $f(0) = 0$ is not even a local extremum.

Here is the picture for purposes of applying the First Derivative Test:

$f'(x) < 0$ $\quad$ $f'(x) > 0$ $\quad$ $f'(x) > 0$ $\quad$ $f'(x) < 0$

-3 $\searrow$ $-\pi/2$ $\nearrow$ 0 $\nearrow$ $\pi/2$ $\searrow$ 3

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13. (10 pts.) Show that the equation $x \cdot \ln(x) = 3$ has exactly one solution in the interval $[2,4]$.

Let $f(x) = x \cdot \ln(x)$ on $[2,4]$. Since f is a product of two continuous functions, f is continuous on $[2,4]$. Recall that the natural logarithm is strictly increasing for $x > 0$ and that $2 < e < 3$. Thus, we have

$$f(2) = 2 \cdot \ln(2) < 2 \cdot \ln(e) = 2 < 3 < 4 = 4 \cdot \ln(e) < 4 \cdot \ln(4) = f(4).$$

Since $f(2) < 3 < f(4)$ and f is continuous on $[2,4]$, the Intermediate Value Theorem implies that there is at least one number c in $(2,4)$ with $f(c) = 3$.

To see that there is no more than one such number, observe that $f'(x) = \ln(x) + 1$, and that if $2 < x < 4$, then we have $0 = \ln(1) < \ln(x)$. Thus, if $2 < x < 4$, then $f'(x) > 0$. It follows that f is strictly increasing on $[2,4]$, since f is continuous on $[2,4]$, and thus, is one-to-one on $[2,4]$. This means that f assumes each value in its range only once.

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14. (10 pts.) Use the first derivative to determine the open intervals where the polynomial function $f(x) = (x - 1)^3(x - 2)^4$ is increasing and decreasing. Be specific.

$$\begin{aligned} f'(x) &= 3(x - 1)^2(x - 2)^4 + 4(x - 1)^3(x - 2)^3 \\ &= (x - 1)^2(x - 2)^3[3(x - 2) + 4(x - 1)] \\ &= (x - 1)^2(x - 2)^3(7x - 10) \\ &= 7(x - 1)^2(x - (10/7))(x - 2)^3 \end{aligned}$$

First observe that the factor $(x - 1)^2$ is positive when it is not zero. Thus, the sign of $f'(x)$ is essentially determined by the remaining factors. Consequently, $f'(x) > 0$ when $x < 1$, or $1 < x < 10/7$, or $2 < x$, and $f'(x) < 0$ when $10/7 < x < 2$. It follows that f is increasing on $(-\infty , 10/7) \cup (2 , \infty)$, and f is decreasing on $(10/7 , 2)$. Here's the big picture:

$f'(x) > 0$		$f'(x) > 0$		$f'(x) < 0$		$f'(x) > 0$
↗	1	↗	10/7	↘	2	↗

Silly Ten Point Bonus: Let $f(\theta) = \sin^\circ(\theta)$ and $g(\theta) = \cos^\circ(\theta)$ be your friendly old sine and cosine functions whose argument θ is measured in *degrees*. Compute $f'(\theta)$ and $g'(\theta)$.

First observe that $f(\theta) = \sin^\circ(\theta) = \sin(\pi\theta/180)$ and that $g(\theta) = \cos^\circ(\theta) = \cos(\pi\theta/180)$, where $\sin(\cdot)$ and $\cos(\cdot)$ are our friendly real number/radian measure versions of sine and cosine with the nice, simple derivatives. Thus, simply using chain rule and the aliases of f and g, we have

$$f'(\theta) = (\pi/180)\cos(\pi\theta/180) = (\pi/180)\cos^\circ(\theta)$$

and

$$g'(\theta) = -(\pi/180)\sin(\pi\theta/180) = -(\pi/180)\sin^\circ(\theta).$$