
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

---

1. (10 pts.) Pretend y is a function of x. Using implicit differentiation, compute dy/dx and d^2y/dx^2 when $y^2 - x^2 = 25$. **Label your expressions correctly or else.**

By pretending y is a function of x and differentiating both sides of the given equation, we obtain $2y\frac{dy}{dx} - 2x = 0$, which implies that $\frac{dy}{dx} = \frac{x}{y}$. Consequently, a second differentiation using quotient rule yields

$$\frac{d^2y}{dx^2} = \left(\frac{x}{y}\right)' = \frac{(1\cdot y) - \left(x\frac{dy}{dy}\right)}{y^2}$$
$$= \frac{y - x(x/y)}{y^2}$$
$$= \frac{y^2 - x^2}{y^3} = \frac{25}{y^3}.$$

---

2. (10 pts.) Compute the first derivative of each of the following functions.

(a) $f(x) = \sec^{-1}(x)$ $f'(x) = \dfrac{1}{|x|(x^2 - 1)^{1/2}}$

(b) $f(x) = \cos^{-1}(x)$ $f'(x) = \dfrac{-1}{(1 - x^2)^{1/2}}$

(c) $f(x) = \cot^{-1}(x)$ $f'(x) = \dfrac{-1}{1 + x^2}$

(d) $f(x) = \sin^{-1}(x)$ $f'(x) = \dfrac{1}{(1 - x^2)^{1/2}}$

(e) $f(x) = \tan^{-1}(x)$ $f'(x) = \dfrac{1}{1 + x^2}$

3. (16 pts.) Fill in the blanks of the following analysis with the correct terminology.

Let $f(x) = 8x^3 - x^4$. Then $f'(x) = -4x^3 + 24x^2 = -4x^2(x - 6)$.

Consequently, $x = 0$ and $x = 6$ are <u>critical</u> points of f. Since $f'(x) < 0$ for $6 < x$, f is <u>decreasing</u> on the set $(6,\infty)$. Also, because $f'(x) > 0$ when $0 < x < 6$ or $x < 0$, and f is continuous, f is <u>increasing</u> on the interval $(-\infty , 6)$. Using the first derivative test, it follows that f has a(n) <u>maximum [global or absolute]</u> at $x = 6$, and <u>neither kind of local extremum</u> at $x = 0$.

Since $f''(x) = -12x^2 + 48x = -12x\cdot(x - 4)$, we have $f''(0) = 0$, $f''(4) = 0$, $f''(x) > 0$ when $0 < x < 4$, and $f''(x) < 0$ when $x > 4$ or $x < 0$. Thus, f is <u>concave up</u> on the interval $(0,4)$, f is <u>concave down</u> on the set $(-\infty ,0) \cup (4,\infty)$, and f has <u>inflection points</u> at $x = 0$ and $x = 4$.

---

4. (4 pts.) Find the function f(x) that satisfies the following two equations: $f'(x) = 4\sec^2(x) + (4/\pi)$ for every real number x in $(-\pi/2,\pi/2)$, and $f(\pi/4) = 16$.

Since $f'(x) = 4\sec^2(x) + (4/\pi)$, we have

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int 4\sec^2(x) + (4/\pi)\,dx \\ &= 4\tan(x) + (4/\pi)x + c \end{aligned}$$

for some real number c. From what we now know about the structure of f, $f(\pi/4) = 16$ implies that

$$16 = 4\tan(\pi/4) + (4/\pi)(\pi/4) + c$$

Solving this little linear equation yields $c = 11$. Thus,

$$f(x) = 4\cdot\tan(x) + (4/\pi)x + 11.$$

5. (10 pts.) (a) Find all the critical points of the function $f(x) = 3(x^2 - 10x)^{1/3}$. (b) Apply the second derivative test at each critical point, c, where $f'(c) = 0$, and draw an appropriate conclusion.

First, $f'(x) = (2x - 10)/(x^2 - 10x)^{2/3} = 2(x - 5)/(x(x - 10))^{2/3}$ for $x \neq 0$ and $x \neq 10$. Therefore, f has critical points at $x = 0$, $x = 5$, and $x = 10$. Only at $x = 5$ do we have $f'(x) = 0$. Observe that we have

$$f''(x) = \frac{[2(x^2 - 10x)^{2/3} - (2x-10)(2/3)(x^2-10x)^{-1/3}(2x-10)]}{(x^2 - 10x)^{4/3}}$$

for $x \neq 0$ and $x \neq 10$. It follows that $f''(5) > 0$. Since $f''(x)$ is continuous in an open interval containing $x = 5$, near $x = 5$ we have $f''(x) > 0$. Consequently, the second derivative test, the weak version found in E&P, implies that f has a relative minimum at $x = 5$. [To see $f''(5) > 0$ easily, note that if $x = 5$, the second term of the numerator of f'' is zero, and what remains is a quotient of squares.]

6. (10 pts.) Evaluate each of the following anti-derivatives.

(a)

$$\int 8x^7 - \frac{1}{x} - \frac{12}{x^3} + 20\cdot\cos(10x)\, dx = x^8 - \ln|x| + 6x^{-2} + 2\cdot\sin(10x) + C$$

(b)

$$\int e^x + e^{-x} - \frac{x^4 - 14x^3 + 14}{x^2} + \frac{1}{1+x^2}\, dx = \int e^x + e^{-x} - (x^2 - 14x + 14x^{-2}) + \frac{1}{1+x^2}\, dx$$

$$= e^x - e^{-x} - (1/3)x^3 + 7x^2 + 14x^{-1} + \tan^{-1}(x) + C$$

Silly Ten Point Bonus: Suppose the generic cubic function $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$ has exactly two distinct real critical points. Must this varmint have an inflection point exactly midway between these critical points? Proof??

The answer to the question is 'yes'.

Since $f'(x) = 3ax^2 + 2bx + c$, f has precisely two real critical points if $D = (2b)^2 - 4(3a)(c)$ is positive, in which case the critical points are $x_1 = [-(2b) + (D)^{1/2}]/(6a)$ and $x_2 = [-(2b) - (D)^{1/2}]/(6a)$.

Plainly $f''(x) = 6ax + 2b = 6a(x - (-b/3a))$.

Since $a \neq 0$, either $a > 0$ or $a < 0$. If $a > 0$, then $f''(x) < 0$ when $x < -b/3a$, and $f''(x) > 0$ when $x > -b/3a$. Thus, f is concave down on $(-\infty,-b/3a)$, concave up on $(-b/3a,\infty)$, and has its only inflection point at $x = -b/3a$. When $a < 0$, the concavity on each interval is reversed, but there is exactly one inflection point at $x = -b/3a$, too.

Finally, observe that $x_3 = -b/3a = (x_1 + x_2)/2$. [Harder: Is $(x_3,f(x_3))$ midway on the line segment in 2-space from $(x_1,f(x_1))$ to $(x_2,f(x_2))$??]

9. (20 pts.) Limits to lament. Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible. For some of these, L'Hopital's Rule may prove useful.

(a)

$$\lim_{x \to \infty} \frac{2x^2 - 1}{5x^2 + 3x} = \lim_{x \to \infty} \frac{2 - x^{-2}}{5 + 3x^{-1}} = \frac{2}{5}$$ without using L'Hopital's rule at all.

(b)

$$\lim_{x \to 0} \frac{\sin(5x)}{\tan(3x)} \overset{(L'H)}{=} \lim_{x \to 0} \frac{5\cos(5x)}{3\sec^2(3x)} = \frac{5}{3}\ .$$

(c)

$$\lim_{x \to \pi/2} \frac{4x - 2\pi}{\tan(2x)} \overset{(L'H)}{=} \lim_{x \to \pi/2} \frac{4}{2\sec^2(2x)} = 2$$

(d)

$$\lim_{x \to 0} \frac{1}{x} \ln\left(\frac{4x + 8}{7x + 8}\right) = \lim_{x \to 0} \frac{\ln(4x + 8) - \ln(7x + 8)}{x}$$

$$\overset{(L'H)}{=} \lim_{x \to 0}\left(\frac{4}{4x + 8} - \frac{7}{7x + 8}\right) = -\frac{3}{8}.$$

(e) Suppose that A > 0 is a real number. Then

$$\lim_{x \to \infty} ((x^2 + Ax)^{1/2} - x) = \lim_{x \to \infty} \frac{(1 + Ax^{-1})^{1/2} - 1}{x^{-1}}$$

$$\overset{(L'H)}{=} \lim_{x \to \infty} \frac{(1/2)(1 + Ax^{-1})^{-1/2}(-Ax^{-2})}{-x^{-2}}$$

$$= \frac{A}{2}.$$

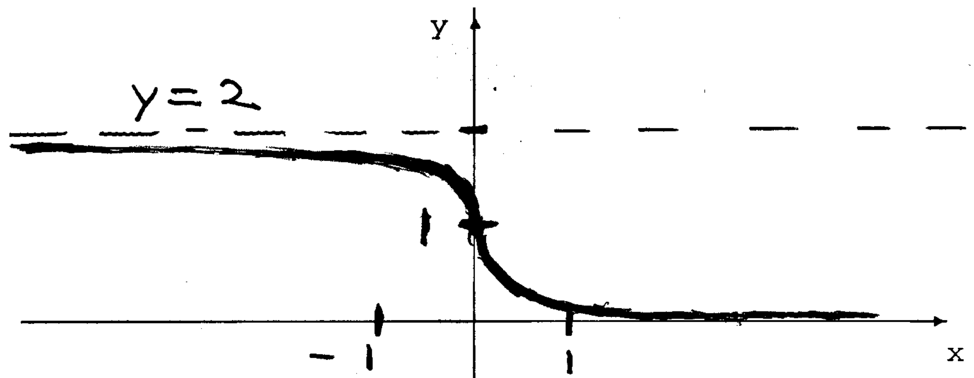
This may also be done without using L'Hopital's Rule. See Spring, 2003, Test 3, Problem 9(a). Merely rationalize the numerator and follow your nose.

---

11. (10 pts.) Very carefully sketch the following function. Use all the data provided and label very carefully.

$f(x)$ is continuous on $\mathbb{R}$ and satisfies the following:

(1) $f(0) = 1$;
(2) $x \neq 0 \Rightarrow f'(x) < 0$;
(3) $\lim_{x \to 0^-} f'(x) = -\infty$ and $\lim_{x \to 0^+} f'(x) = -\infty$;
(4) $x < 0 \Rightarrow f''(x) < 0$ and $x > 0 \Rightarrow f''(x) > 0$
(5) $f(x) \longrightarrow 2$ as $x \longrightarrow -\infty$, and $f(x) \longrightarrow 0$ as $x \longrightarrow \infty$.



(5) yields horizontal asymptotes.
(4) => concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

(2) => always decreasing
(3) => vertical tangent @ (0,1)

---

12. (10 pts.)
Determine where the function $f(x) = x^4 - 6x^2$ is concave up, concave down, and locate any inflection points it may have.

$f'(x) = 4x^3 - 12x$ and $f''(x) = 12x^2 - 12 = 12(x -(-1))(x - 1)$ It follows that if $x < -1$ or $x > 1$, then $f''(x) > 0$, and that if we have $-1 < x < 1$, then $f''(x) < 0$. Consequently, f is concave up on $(-\infty, -1) \cup (1 , \infty)$, concave down on the interval $(-1 , 1)$, and has inflection points at $x = -1$ and at $x = 1$.

Silly Ten Point Bonus: Suppose the generic cubic function $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$ has exactly two distinct real critical points. Must this varmint have an inflection point exactly midway between these critical points? Proof?? [Details!! Work on the back of Page 4. You do not have room here.] **[This is on the bottom of Page 3 of 5.]**