
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

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1. (10 pts.) Using the text's four step process, show completely how to obtain the slope-predictor function m(x) for the function $f(x) = x^3$.

$$m(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2.$$

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2. (15 pts.) Let $f(x) = 2x(x - 4)$.

(a) Then the slope-predictor function for f is given by

$m(x) = 4x - 8$ since $f(x) = 2x^2 - 8x$.

(b) It turns out that the graph of f has a horizontal tangent line at precisely one point on the graph of f, $(x_1,f(x_1))$. What is this ordered pair?

Plainly f has a horizontal tangent line at $(x_1,f(x_1))$ precisely when the slope-predictor function of f at x_1 is zero. Now

$$m(x_1) = 0 \Leftrightarrow 4x_1 - 8 = 0 \Leftrightarrow x_1 = 2.$$

Thus, computing the value of $f(x_1)$, we have

$$(x_1,f(x_1)) = (2, -8).$$

$(x_1,f(x_1)) =$ (2, -8).

(c) What is the slope of the line perpendicular to the line that is tangent to the graph of f at the point P(4,0) ?? Thus, find the slope of the line normal to the graph at P(4,0).

Since the slope-predictor function is $m(x) = 4x - 8$, the slope of the line tangent to the graph at (4,0) is $m(4) = 8$. Thus, the slope of the normal line is $m = -1/8$.

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3. (25 pts.) For each of the following, find the limit if the limit exists. If the limit fails to exist, say so. Be as precise as possible here. [Work on the back of Page 1 of 4 if you run out of room here.]

(a) $\lim_{x\to 2}\frac{x^2-x-2}{x^2-4} = \lim_{x\to 2}\frac{(x-2)(x+1)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{x+1}{x+2} = \frac{3}{4}$

(b) $\lim_{x\to 9}\frac{x - 9}{3 - x^{1/2}} = \lim_{x\to 9}\frac{(x^{1/2}-3)(x^{1/2}+3)}{3-x^{1/2}} = \lim_{x\to 9}(-1)\cdot(x^{1/2}+3) = -6$

(c)
$$\lim_{x\to 0}\frac{x\cdot\sin(2x)}{1 - \cos(x)} = \lim_{x\to 0}\frac{x\cdot 2\cdot\sin(x)\cdot\cos(x)\cdot(1+\cos(x))}{\sin^2(x)}$$
$$= \lim_{x\to 0}\frac{2x}{\sin(x)}\cdot\lim_{x\to 0}\cos(x)\cdot\lim_{x\to 0}(1+\cos(x))$$
$$= 4$$

(d) $\lim_{x\to 5^-}\frac{2x - 10}{|x - 5|} = \lim_{x\to 5^-}\frac{2(x - 5)}{-(x - 5)} = -2$

(e) $\lim_{x\to 1}(4x + 5)^{1/2} = 9^{1/2} = 3$

Answer for Problem 10: Give a complete ε - δ proof that

$$\lim_{x\to -3}(7x - 9) = -30 .$$

Proof: Let ε > 0 be arbitrary. Set δ = ε/7. Observe that δ > 0. Suppose now that x satisfies $0 < |x - (-3)| < \delta$. We shall now verify that $0 < |x - (-3)| < \delta$ implies $|(7x - 9) - (-30)| < \varepsilon$. Now

$$\begin{aligned} 0 < |x - (-3)| < \delta \;&\Rightarrow\; |x + 3| < \varepsilon/7 \\ &\Rightarrow\; 7|x + 3| < \varepsilon \\ &\Rightarrow\; |7x + 21| < \varepsilon \\ &\Rightarrow\; |(7x - 9) - (-30)| < \varepsilon. \end{aligned}$$

Since, given an arbitrary ε > 0, we have produced a number δ > 0 such that, if x satisfies $0 < |x - (-3)| < \delta$, then $|(7x-9) - (-30)| < \varepsilon$, we have proved (7x-9) —> -30 as x —> -3.//

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10 Point Bonus: Suppose $x_0 > 0$. Give an ε - δ proof that

$\lim_{x\to x_0} x^{1/2} = (x_0)^{1/2}$ // Proof: Let ε > 0 be arbitrary. Set

$\delta = (x_0)^{1/2}\varepsilon$. We shall show that this δ does the job. Suppose now that x is a nonnegative number such that $0 < |x - x_0| < \delta$. Then we have

$$|x^{1/2} - (x_0)^{1/2}| = \frac{|x - x_0|}{|x^{1/2} + x_0^{1/2}|} \le \frac{|x - x_0|}{x_0^{1/2}} < \frac{\delta}{x_0^{1/2}} = \varepsilon.$$

[Working with the "conclusion", rationalize the numerator.

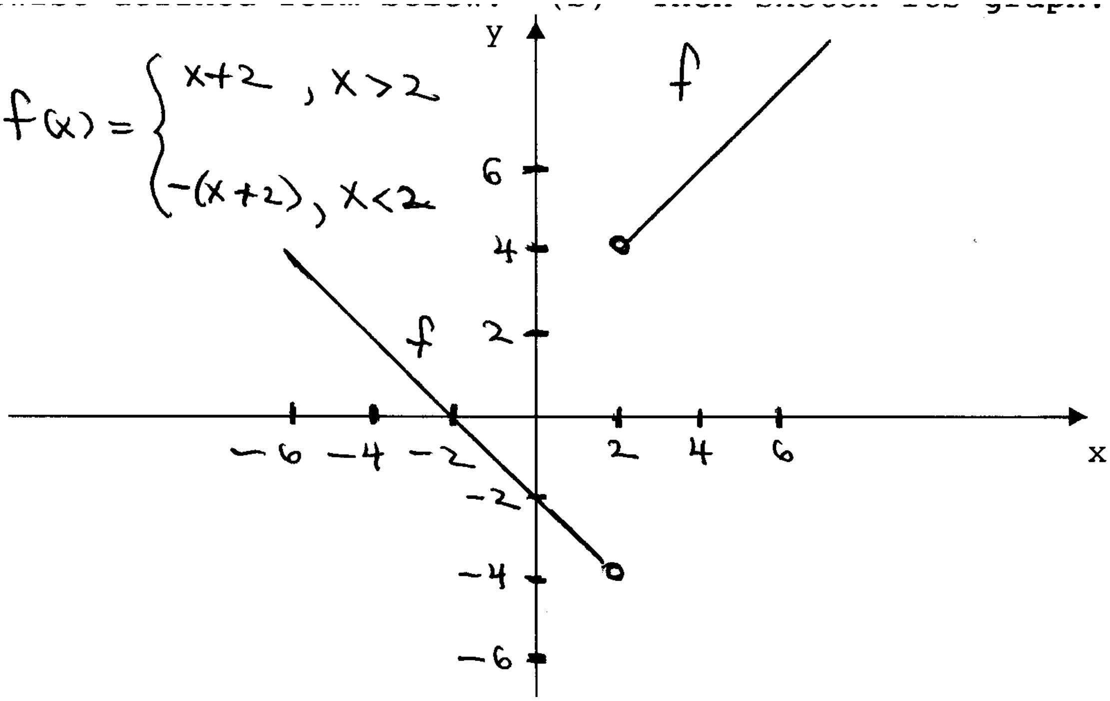
Plainly, $0 < x_0^{1/2} \le x_0^{1/2} + x^{1/2} \Leftrightarrow \frac{1}{x_0^{1/2} + x^{1/2}} \le \frac{1}{x_0^{1/2}}$ when $0 \le x$.]

4. (10 pts.) (a) First write the function

$$f(x) = \frac{x^2 - 4}{|x - 2|}$$

in a piecewise-defined form below. (b) Then sketch its graph.

$$f(x) = \begin{cases} x+2, & x > 2 \\ -(x+2), & x < 2 \end{cases}$$



5. (15 pts.) Suppose that

$$h(x) = \begin{cases} x^2 - 2x, & \text{if } x > 1 \\ 2, & \text{if } x = 1 \\ 2x - 2, & \text{if } x < 1 \end{cases}$$

Evaluate each of the following limits.

(a) $\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} (x^2 - 2x) = -1$

(b) $\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} (2x - 2) = 0$

(c) $\lim_{x \to 2} h(x) = \lim_{x \to 2} (x^2 - 2x) = 0$

(d) $\lim_{x \to -2} h(x) = \lim_{x \to -2} (2x - 2) = -6$

(e) $\lim_{x \to 1} h(x) =$ This limit does not exist. Look at (a) and (b). The two one-sided limits at x = 1 exist and have different values.

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6. (5 pts.) Using complete sentences and appropriate notation, provide the precise ε - δ mathematical definition of

$$\lim_{x\to a} f(x) = L$$

Suppose that f is a function that is defined everywhere in some open interval containing x = a, except possibly at x = a. We write

$$\lim_{x\to a} f(x) = L$$

if L is a number such that for each ε > 0, we can find a δ > 0, such that if x is in the domain of f and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

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7. (5 pts.) Show how to use the squeeze law of limits to provide an evaluation of the following limit that is completely correct. You will need to show how to build a suitable inequality to provide a complete solution.

$\lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0$ Here's why: For $x \neq 0$, $-1 \leq \sin(1/x) \leq 1$ implies that $-x^2 \leq x^2\sin(1/x) \leq x^2$. Since $x^2 \to 0$ as $x \to 0$, the squeezing theorem implies $x^2\sin(1/x) \to 0$ as $x \to 0$.

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8. (5 pts.) It turns out that the slope-predictor function for the function f(x) = sec(x) is the function m(x) = sec(x)tan(x). Use this to obtain an equation for the line tangent to the graph of f(x) = sec(x) at $x_0 = \pi/4$.// Since $f(\pi/4) = \sec(\pi/4) = 2^{1/2}$ and $m(\pi/4) = \sec(\pi/4)\tan(\pi/4) = 2^{1/2}$, an equation for the line tangent to the graph of f at $x_0 = \pi/4$ is
$y - 2^{1/2} = 2^{1/2}(x - \pi/4)$. There are, of course, infinitely many equations equivalent to this one.

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9. (5 pts.) The line tangent to the graph of $y = x^2$ at $x = x_0$ is given by the equation

$$y-(x_0)^2 = 2x_0(x-x_0) \ .$$

Where does this line intersect the x-axis?? [Be CAREFUL. There is a trap.] The line above intersects the x-axis when y = 0. Substituting, this happens if x satisfies the equation

$$-(x_0)^2 = 2x_0(x-x_0) \text{ which is equivalent to } (x_0)^2 = 2x_0x.$$

If $x_0 \neq 0$, this happens at $x = x_0/2$. So the point of intersection is $(x_0/2,0)$. On the other hand, when $x_0 = 0$, every real number, x, satisfies the equation. In this case, every point of the x-axis is a point of intersection. Of course, since the "tangent line" is the x-axis, this is no surprise.

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10. (5 pts.) On the back of Page 3 of 4, give a complete ε - δ proof that $\lim_{x\to -3}(7x - 9) = -30$. [Answer on Page 2 of 4.]

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10 Point Bonus: Suppose $x_0 > 0$. Give an ε - δ proof that $\lim_{x\to x_0} x^{1/2} = (x_0)^{1/2}$ on the back of Page 2 of 4. [Answer: Page 2]