Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (10 pts.) (a) Using a complete sentence and appropriate notation, provide the precise mathematical definition of continuity of a function f(x) at a point x = a.//

A function f is continuous at x = a if $\lim_{x \to a} f(x) = f(a)$.

(b) Explicitly using the definition of continuity, find all values for the constant c, if possible, that will make the function f(x) defined below continuous at x = 0 . Suppose

$$f(x) = \begin{cases} c^2 \cdot \cos(x/2) & , \quad x \geq 0 \\ \sin(25x)/x & , \quad x < 0. \end{cases}$$

Plainly, from the definition, f above is continuous at x = 0 if, and only if $\lim_{x \to 0} f(x) = f(0) = c^2$. Since

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin(25x)}{x} = \lim_{x \to 0^-} \frac{25\sin(25x)}{25x} = 25 ,$$

and the right-handed limit at 0 exists and has value c^2, f is continuous at x = 0 if, and only if $c^2 = 25$. Thus c = 5 or c = -5 will work.

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2. (15 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative, f′(x), of a function f(x).

The function f′ defined by the equation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the derivative of f with respect to x. The domain of f′ consists of all x in the domain of f for which the limit above exists.

(b) Using only the definition of the derivative as a limit, show all steps of the computation of f′(x) when $f(x) = (2x + 1)^{1/2}$.

If x > -1/2, then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(2(x+h)+1)^{1/2} - (2x+1)^{1/2}}{h} \\ &= \lim_{h \to 0} \frac{2h}{h[(2(x+h)+1)^{1/2} + (2x+1)^{1/2}]} = \frac{1}{(2x+1)^{1/2}}. \end{aligned}$$

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3. (10 pts.) Locate and determine the maximum and minimum values of the function $f(x) = x + 4x^{-1}$ on the interval $[-4, -1]$. What magic theorem allows you to conclude that f(x) has a maximum and minimum even before you attempt to locate them? Why??

The rational function f is continuous on the interval $[-4, -1]$. Consequently, the magical Extreme Value Theorem, or what E&P call the Absolute Maxima and Minima Theorem, guarantees that f has absolute extrema of both flavors on $[-4, -1]$. Since

$$f'(x) = 1 - 4x^{-2} = \frac{x^2 - 4}{x^2} = \frac{(x-2)(x+2)}{x^2},$$

f has a single critical point at $x = -2$. Since $f(-4) = -5$, $f(-2) = -4$, and $f(-1) = -5$, the maximum is -4 and occurs at $x = -2$, and the minimum is -5 and occurs at $x = -4$ and at $x = -1$.

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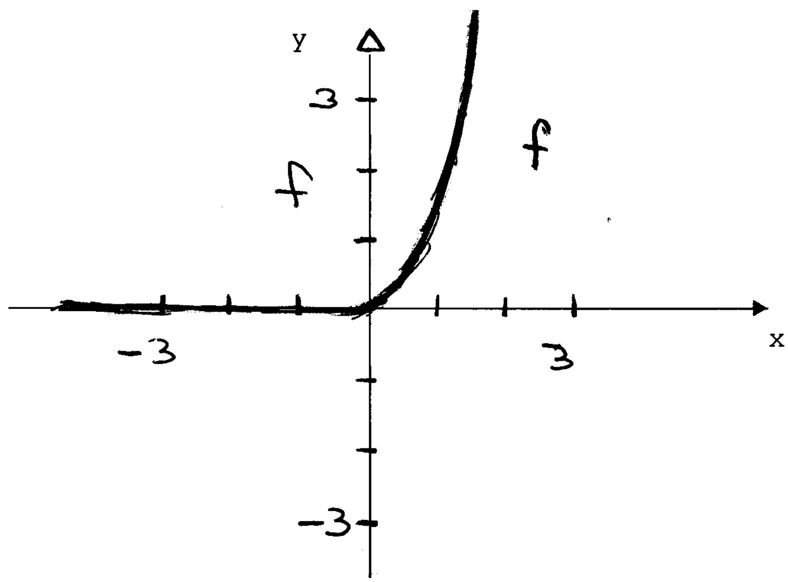
4. (15 pts.) (a) Sketch the graph of the function $f(x) = |x|^3 + x^3$. [Hint: First write f in a piecewise-defined form.] (b) Find the derivative for $x > 0$ and for $x < 0$. [These are two separate cases.] (c) Show the computation of the one-sided derivatives at $x = 0$. What can you conclude from this??

(a)



$$f(x) = \begin{cases} x^3+x^3, & \text{if } x \ge 0 \\ -x^3+x^3, & \text{if } x < 0 \end{cases} = \begin{cases} 2x^3, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases}$$

(b) If $x > 0$, then $f'(x) = 6x^2$, and if $x < 0$, then $f'(x) = 0$.

(c) $f'_-(0) = \lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{0-0}{h} = 0$ and

$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^+} \frac{2h^3-0}{h} = 0$ imply that $f'(0) = 0$.

5. (25 pts.) Obtain the derivative of each of the following functions. You may use any of the rules of differentiation at your disposal. Do not simplify the algebra. [5 pts./part]

(a) $f(x) = 7x^{-3} - 3x^{5} + 20x^{4/5}$

$f'(x) = -21x^{-4} - 15x^{4} + 16x^{-1/5}$

(b) $g(x) = (x^{5} - 7x^{-1})(25x^{2} + 6x^{-5})$

$g'(x) = (x^{5} - 7x^{-1})(50x - 30x^{-6}) + (5x^{4} + 7x^{-2})(25x^{2} + 6x^{-5})$

(c) $h(t) = \dfrac{3t^{2} + 7t}{2t^{4} - t^{3}}$

$h'(t) = \dfrac{(6t+7)(2t^{4}-t^{3}) - (3t^{2}+7t)(8t^{3}-3t^{2})}{(2t^{4} - t^{3})^{2}}$

(d) $L(x) = (x^{4} + x^{-2})^{8/5}$

$L'(x) = (8/5)(x^{4} + x^{-2})^{3/5}(4x^{3} - 2x^{-3})$

(e) $y = (x^{2} - 2x^{-2})^{25}(10x^{3/2} + 10x)^{5}$

$$\frac{dy}{dx} = 25(x^{2}-2x^{-2})^{24}(2x+4x^{-3})(10x^{3/2}+10x)^{5} + (x^{2}-2x^{-2})^{25}(5)(10x^{3/2}+10x)^{4}(15x^{1/2}+10)$$

6. (10 pts.) (a) Using complete sentences and appropriate notation, state the theorem that is concerned with the intermediate value property of continuous functions.

Suppose that f is continuous on a closed interval [a,b]. If K is any number between f(a) and f(b), then there exists at least one number c in (a,b) such that f(c) = K.

(b) Apply the theorem concerning the intermediate value property of continuous functions to show that the given equation has a solution in the given interval.

$$x^5 - 6x^3 + 3 = 0 \text{ on } [-1,1]$$

Explain completely. Deal with all the magical hypotheses.

Let $f(x) = x^5 - 6x^3 + 3$ on [-1,1]. Observe that f(-1) = 8, f(1) = -2, and K = 0 is a number between f(-1) and f(1). Since f is a polynomial, f is continuous on the interval [-1,1]. Consequently, the function f satisfies the hypotheses of the theorem that is concerned with the intermediate value property of continuous functions on the interval [-1,1]. Thus, we are entitled to invoke the magical conclusion that asserts that there is at least one number c in (-1,1) where f(c) = 0.

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7. (5 pts.) Give an example of a continuous function f(x) which has a point c in its domain with f′(c) = 0, but such that f(c) is not a local extreme value.

Perhaps the simplest example of such a function is $f(x) = x^3$ which has a single critical point at c = 0 with f(0) = 0 neither a local maximum nor a local minimum. That 0 is neither is clear since $x < 0$ implies that $x^3 < 0$, and $0 < x$ implies that $0 < x^3$.

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8. (10 pts.) Find all points in the domain of

$$f(x) = x - 3x^{1/3}$$

where the tangent line is either horizontal or vertical, and clearly say which points have horizontal tangent lines and which have vertical tangent lines.

$$f'(x) = 1 - x^{-2/3} = \frac{x^{2/3} - 1}{x^{2/3}} = \frac{(x^{1/3}-1)(x^{1/3}+1)}{x^{2/3}}, \text{ for } x \neq 0.$$

Plainly the derivative of f is zero if, and only if x = 1 or x = -1. Thus, we have horizontal tangent lines at x = 1 or at x =-1. Evidently, since zero is in the domain of f but not in the domain of f′, we should examine the limit behavior of $|f'(x)|$ as $x \to 0$. Since

$$\lim_{x\to 0} |f'(x)| = \lim_{x\to 0} \left|\frac{x^{2/3}-1}{x^{2/3}}\right| = \infty ,$$

there is a vertical tangent line at x = 0. So (0, f(0)) = (0,0) is a point on the graph of f with a vertical tangent line.