
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (5 pts.) Use logarithmic differentiation to find dy/dx when $y = x^{\sec(x)}$. **Label your expressions correctly or else.**

$$\ln(y) = \sec(x)\ln(x) \Rightarrow \frac{1}{y}\frac{dy}{dx} = \sec(x)\tan(x)\ln(x) + \frac{1}{x}\sec(x)$$
$$\Rightarrow \frac{dy}{dx} = \left(\sec(x)\tan(x)\ln(x) + \frac{1}{x}\sec(x)\right)x^{\sec(x)}.$$

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2. (5 pts.) Use a linear approximation L(x) to an appropriate function f(x), with an appropriate value of a, to estimate the value of $80^{3/4}$.

Let $f(x) = x^{3/4}$ and $a = 81 = 3^4$. Then $f'(x) = (3/4)x^{-1/4}$. So

$$L(x) = f(a) + f'(a)(x - a)$$
$$= (3^4)^{3/4} + (3/4)(3^4)^{-1/4}(x - 81)$$
$$= 27 + (1/4)(x - 81).$$

Hence, $80^{3/4} \approx L(80) = 27 - (1/4) = 107/4$.

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3. (5 pts.) Write dy in terms of x and dx when $y = \sin(2x)e^{-3x}$.

$$dy = (2\cos(2x)e^{-3x} - 3\sin(2x)e^{-3x})dx$$

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4. (5 pts.) Find all points in the interval $[0,2\pi]$ where the graph of $f(x) = \sqrt{2}x + 2\cos(x)$ has a horizontal tangent line.

$f'(x) = \sqrt{2} - 2\sin(x)$. So $f'(x) = 0 \Leftrightarrow \sin(x) = \frac{\sqrt{2}}{2}$. Thus, f has a horizontal tangent line at $x_0 = \pi/4$ and at $x_1 = 3\pi/4$.

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5. (5 pts.) Find the function f(x) that satisfies the following two equations: $f'(x) = \frac{3}{2}x^{1/2}$ *and* $f(1) = 10$.

Clearly $g(x) = x^{3/2}$ satisfies $g'(x) = (3/2)x^{1/2}$. Since f and g have the same derivative on $(0,\infty)$, $f(x) - g(x) = K$ for some number K. Since $9 = 10 - 1 = f(1) - g(1) = K$,

$$f(x) = g(x) + 9 = x^{3/2} + 9.$$

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6. (10 pts.) (a) Using implicit differentiation, compute dy/dx and when $x^4 + y^4 = 17$. **Label your expressions correctly or else.**

Pretend that y is a function of x. Then by differentiating both sides of the equation, we have

$$4x^3 + 4y^3\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^3}{y^3}.$$

(b) Obtain an equation for the line tangent to the graph of $x^4 + y^4 = 17$ at the point (1,-2).

Since

$$\frac{dy}{dx}\Big|_{(1,-2)} = -\frac{1}{(-2)^3} = \frac{1}{8},$$

an equation for the tangent line is y - (-2) = (1/8)(x - 1) or y = (1/8)x - (17/8).

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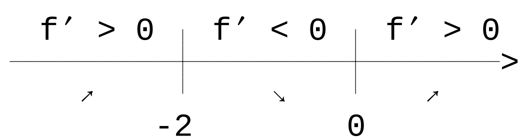
7. (10 pts.) Apply the first derivative test to classify the critical points of the function $f(x) = x^2e^x$. Reveal all the details of your analysis.

Since $f'(x) = x^2e^x + 2xe^x = x(x + 2)e^x$, the critical points of f are x = 0 and x = -2. Clearly the sign of f′ is determined by the factor x(x + 2). Thus, if x < -2, then f′(x) > 0; if -2 < x < 0, then f′(x) < 0; and if x > 0, then f′(x) > 0. It follows from the 1st derivative test that f(-2) is a local maximum and f(0) = 0 is a local minimum. Since f(x) ≥ 0 for each real number x, f(0) is the absolute minimum value of f on the real line. We can see, however, that f(2) cannot be an absolute maximum since $x^2 \le x^2e^x = f(x)$ for each x ≥ 1. Here's a diagram of the first derivative situation:



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8. (5 pts) Rolle's Theorem states that if f(x) is continuous on [a,b] with f(a) = f(b) = 0 and differentiable on (a,b), then there is a number c in (a,b) such that f′(c) = 0. **Give an example of a function f(x) defined on [-1,1] with f differentiable on (-1,1) and f(-1) = f(1) = 0 but such that there is no number c in (-1,1) with f′(c) = 0.** [Hint: Which hypothesis above must you violate??]

Define f on [-1,1] by the formula

$$f(x) = \begin{cases} 0\ , & if\ x = -1\ or\ x = 1, \\ x\ , & if\ -1 < x < 1 \end{cases}$$

Then f′(x) = 1 for x in (-1,1), and f(-1) = f(1) = 0, but there is no number c in the interval (-1,1) with f′(c) = 0. This reveals that continuity on the interval is an essential hypothesis of Rolle's Theorem. Of course this is a property that f cannot have.

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9. (10 pts.) (a) State the Mean Value Theorem of Differential Calculus. Use a complete sentence and appropriate notation.

If f is a function that is continuous on [a,b] and differentiable on (a,b), then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) Show how to use the Mean Value Theorem to prove the following: Suppose that a and b are real numbers with a < b. If f is a function that is continuous on a closed interval [a,b] and is differentiable on the open interval (a,b) with f′(x) < 0 for every x in (a,b), then f is a decreasing function on [a,b].

Proof: Let x_1 and x_2 be arbitrary numbers in [a,b] with $x_1 < x_2$. Then f satisfies the hypotheses of the Mean Value Theorem on $[x_1,x_2]$. Thus, there is a number c in (x_1,x_2) with $(x_2 - x_1)f'(c) = f(x_2) - f(x_1)$. Since $f'(c) < 0$, $f(x_2) - f(x_1) < 0$, implying $f(x_2) < f(x_1)$. Since we have shown that $a \le x_1 < x_2 \le b \Rightarrow f(x_1) > f(x_2)$ for any x_1 and x_2, f is decreasing on [a,b].//[Corollary 3, Chapter 4.3, done in class.]

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10. (10 pts.) A circular oil slick of uniform thickness is caused by a spill of 1 cubic meter of oil. The thickness of the slick is decreasing at the rate of .001 meter/hour. At what rate is the radius increasing when the radius is 8 meters. // Let t_0 denote the point in time, in hours, when the radius reached 8 meters. Now $V(t) = 2\pi r^2(t)h(t)$. Plainly $V'(t) = 0$ and $h'(t) = -1/1000$. Also $V'(t) = 2\pi r(t)r'(t)h(t) + \pi r^2(t)h'(t)$ for each t. Since $1 = \pi r^2(t_0)h(t_0)$ and $r(t_0) = 8$, $h(t_0) = 1/(64\pi)$. Thus, after the algebraic and arithmetical dust settles, we have

$$r'(t_0) = -\frac{r(t_0)h'(t_0)}{2h(t_0)} = \left(-\frac{1}{2}\right)\left[\frac{8}{\left(\frac{1}{64\pi}\right)}\right]\left(-\frac{1}{1000}\right)$$

$$= \frac{32\pi}{125}\ \textit{meters per hour}.$$

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11. (5 pts.) Can there be two numbers x_1 and x_2 in the interval [-1,1] where sin(x) = 3x - 1?? Explain briefly. [Hint: Consider the function f(x) = sin(x) - (3x - 1) on [-1,1].] // No. Let f(x) = sin(x) - (3x - 1) on [-1,1]. Observe that sin(x) = 3x - 1 if, and only if f(x) = sin(x) - (3x - 1) = 0. Thus, the question is equivalent to that of the uniqueness of the zeros of f in the interval [-1,1]. Since $f'(x) = \cos(x) - 3$ and $\cos(x) \le 1$ on (-1,1), we have $f'(x) = \cos(x) - 3 \le 1 - 3 = -2$ on (-1,1). Thus, f is strictly decreasing, and hence one-to-one on [-1,1]. So there is at most one point in [-1,1] where f(x) = 0, or equivalently, sin(x) = 3x - 1.// Note: You may also argue indirectly by using Rolle's Theorem. How? When is cos(x) = 3??

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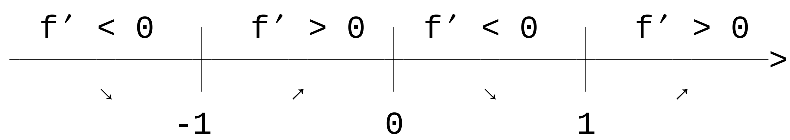
12. (10 pts.) Use the first derivative to determine the open intervals where the polynomial function $f(x) = x^4 - 2x^2$ is increasing and those intervals where f is decreasing. Be specific.

Since $f'(x) = 4x^3 - 4x = 4x(x - (-1))(x - 1)$, it is easy to see that if $x < -1$, then $f'(x) < 0$; if $-1 < x < 0$, then $f'(x) > 0$; if $0 < x < 1$, then $f'(x) < 0$; and if $1 < x$, then $f'(x) > 0$. It follows then that f is decreasing on the open intervals $(-\infty,-1)$ and $(0,1)$, and f is increasing on the open intervals $(-1,0)$ and $(1,\infty)$. A diagram of the sign situation for the first derivative is below:



Where's the no trespassing sign??

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13. (10 pts.) What point on the line $2x + y = 24$ is nearest the origin, the point (0,0)?? [Hint: The square of the distance between points in the plane is easier to deal with than the distance and can tell you which point is closest.]

Let D denote the square of the distance from a point (x,y) on the line defined by $y = 24 - 2x$ to the origin. Then we have $D = (x - 0)^2 + (y - 0)^2 = x^2 + (24 - 2x)^2$. Set

$$D(x) = x^2 + (24 - 2x)^2$$

for each real number x. The point (x_0,y_0) on the line that is nearest the origin has x-component x_0 with $D(x_0)$ an absolute minimum. Now

$$\begin{aligned} D'(x) &= 2x + 2(24 - 2x)(-2) \\ &= 2(x + (4x - 48)) \\ &= 2(5x - 48) \\ &= 10(x - (48/5)). \end{aligned}$$

Since $D'(x) < 0$ for $x < 48/5$ and $D'(x) > 0$ for $x > 48/5$, the first derivative test for absolute extrema tells us that we have an absolute minimum at $x_0 = 48/5$. Consequently,

$$(x_0,y_0) = (48/5,\ 24/5)$$

is the point on the line defined by $2x + y = 24$ that is nearest the origin, (0,0).

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14. (5 points) Would you know a derivative if it hit you with a two by four? What is the value of the following limit??

$$\lim_{h \to 0} \frac{\cos(\frac{\pi}{2}+h) - \cos(\frac{\pi}{2})}{h} = \cos'(\frac{\pi}{2}) = -\sin(\frac{\pi}{2}) = -1.$$