
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (10 pts.) Compute the first derivative of each of the following functions.

(a) $f(x) = \cos^{-1}(x)$ $\qquad f'(x) = \dfrac{-1}{(1-x^2)^{1/2}}$

(b) $f(x) = \sec^{-1}(x)$ $\qquad f'(x) = \dfrac{1}{|x|(x^2-1)^{1/2}}$

(c) $f(x) = \sin^{-1}(x)$ $\qquad f'(x) = \dfrac{1}{(1-x^2)^{1/2}}$

(d) $f(x) = \cot^{-1}(x)$ $\qquad f'(x) = \dfrac{-1}{1+x^2}$

(e) $f(x) = \csc^{-1}(x)$ $\qquad f'(x) = \dfrac{-1}{|x|(x^2-1)^{1/2}}$

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2. (10 pts.) Pretend y is a function of x. Using implicit differentiation, compute dy/dx and d^2y/dx^2 when $y^3 + x^2 + x = 5$. **Label your expressions correctly or else.**

By differenting both sides of $y^3 + x^2 + x = 5$, we obtain $3y^2\dfrac{dy}{dx} + 2x + 1 = 0$, which implies that $\dfrac{dy}{dx} = -\dfrac{2x + 1}{3y^2}$. Thus,

$$\frac{d^2y}{dx^2} = -\left[\frac{2(3y^2) - (2x + 1)(6y\frac{dy}{dx})}{9y^4}\right]$$

$$= -\left[\frac{6y^2 + (2x + 1)\frac{6y(2x+1)}{3y^2}}{9y^4}\right]$$

$$= -\left[\frac{18y^4 + 6y(2x + 1)^2}{27y^6}\right].$$

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3. (10 pts.) (a) Find all the critical points of the function

$$f(x) = 3(x^2 - 4x)^{1/3}$$

(b) Apply the second derivative test at each critical point, c, where f′(c) = 0, and draw an appropriate conclusion.

$$f'(x) = (x^2 - 4x)^{-2/3}(2x-4) = \frac{2(x-2)}{(x^2-4x)^{2/3}}$$

for $x \neq 0$ and $x \neq 4$. Thus, f has as critical points $x = 0$, $x = 2$, and $x = 4$. Note, however, that only at $x = 2$ is the derivative zero. Since

$$f''(x) = \frac{2(x^2-4x)^{2/3} - 2(x-2)(2/3)(x^2-4x)^{-1/3}(2x-4)}{(x^2-4x)^{4/3}},$$

$$f''(2) = \frac{2(-4)^{2/3}}{(-4)^{4/3}} > 0.$$

Since the second derivative is continuous in an interval containing x = 2, the second derivative must be positive in a possibly smaller open interval containing x = 2. Thus, the second derivative test, the weak E&P version, implies f(2) is a local or relative minimum value.

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4. (10 pts.) Evaluate each of the following anti-derivatives.

(a)

$$\int 10x^9 + \frac{3}{x} - \frac{8}{x^5} + 6\sin(3x)\,dx = 10\int x^9 dx + 3\int \frac{1}{x}dx - 8\int x^{-5}dx + 6\int \sin(3x)\,dx$$

$$= x^{10} + 3\ln|x| + 2x^{-4} - 2\cos(3x) + C$$

(b)

$$\int 3e^x - \frac{x^4 + x^3 + 1}{x^2} - \frac{1}{(1-x^2)^{1/2}}\,dx = 3e^x - \frac{x^3}{3} - \frac{x^2}{2} + x^{-1} - \sin^{-1}(x) + C$$

since

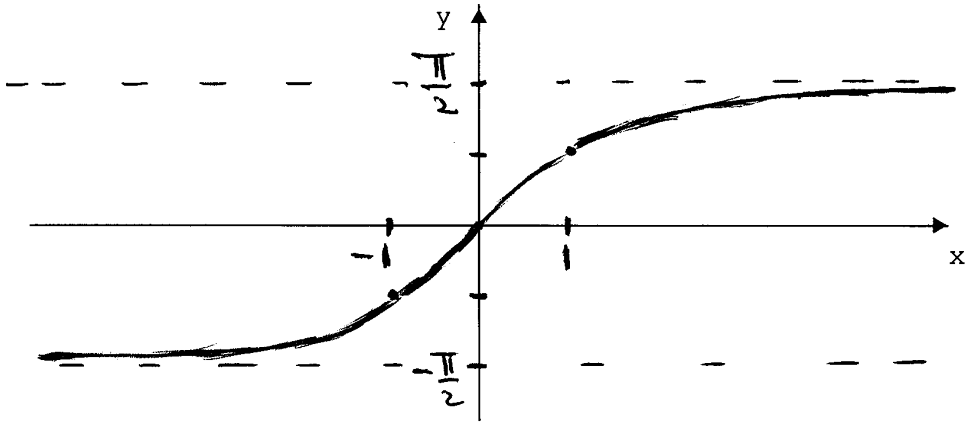
$$\int \frac{x^4 + x^3 + 1}{x^2}\,dx = \int x^2 + x + x^{-2}\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - x^{-1} + C$$

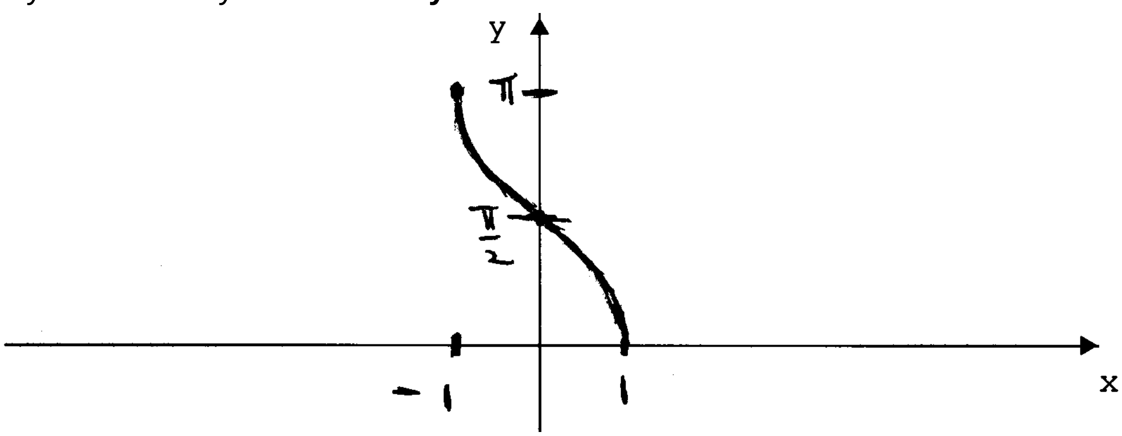
and

$$\int \frac{1}{(1-x^2)^{1/2}}\,dx = \sin^{-1}(x) + C.$$

5. (10 pts.) (a) Carefully sketch the graph of $y = \tan^{-1}(x)$ below. Label very carefully. **This may be found in Section 6.8.**



(b) Carefully sketch the graph of $y = \cos^{-1}(x)$ below. Label very carefully. **This may be found in Section 6.8.**



6. (10 pts.) (a) (5 pts.) Find a function g(x) so that g satisfies the following equation:

$$\int g(x)\,dx = \sec^2(x) + e^{2x} + 2x^5 + C$$

$$g(x) = \frac{d}{dx}\left(\sec^2(x) + e^{2x} + 2x^5 + C\right)$$

$$= 2\sec(x)\sec(x)\tan(x) + 2e^{2x} + 10x^4$$

$$= 2\sec^2(x)\tan(x) + 2e^{2x} + 10x^4.$$

(b) Solve the following initial value problem:

$$\frac{dy}{dx} = 6e^{2x}\ ;\ \ y(0) = 10.$$

First, $\frac{dy}{dx} = 6e^{2x} \Rightarrow y(x) = \int 6e^{2x}dx = 3e^{2x} + C.$

$y(x) = 3e^{2x} + C$ *and* $y(0) = 10 \Rightarrow C = 7.$ *Thus*, $y(x) = 3e^{2x} + 7.$

9. (20 pts.) Limits to lament. Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible. For some of these, L'Hopital's Rule may prove useful.

(a) $$\lim_{x \to 0} \frac{\sin(4x)}{\tan(5x)} \overset{(L'H)}{=} \lim_{x \to 0} \frac{4\cos(4x)}{5\sec^2(5x)} = \frac{4}{5}.$$

(b) $$\lim_{x \to \infty} \frac{2x^3 - 1}{5x^2 + 3x^3} = \lim_{x \to \infty} \frac{2 - \dfrac{1}{x^3}}{\dfrac{5}{x} + 3} = \frac{2}{3}.$$

without using L'Hopital's Rule. You may, of course, apply L'Hopital's Rule a couple of times to obtain the same numerical result.

(c)

$$\lim_{x \to \infty} ((x^2 + 2x)^{1/2} - (x^2 - 2x)^{1/2}) = \lim_{x \to \infty} \frac{4x}{(x^2 + 2x)^{1/2} + (x^2 - 2x)^{1/2}}$$
$$= \lim_{x \to \infty} \frac{4}{\left(1 + \dfrac{2}{x}\right)^{1/2} + \left(1 - \dfrac{2}{x}\right)^{1/2}} = 2.$$

without using L'Hopital's Rule by rationalizing the numerator, etc. You may also evaluate this limit by using L'Hopital's Rule after doing some routine algebra at the beginning, but it actually involves more work than the "tricky algebra" here.

(d) $$\lim_{x \to \pi/2} \frac{8x - 4\pi}{\tan(2x)} \overset{(L'H)}{=} \lim_{x \to \pi/2} \frac{8}{2\sec^2(2x)} = 4.$$

(e) $$\lim_{x \to 0} \frac{1}{x} \ln\left(\frac{4x + 8}{7x + 8}\right) = \lim_{x \to 0} \frac{\ln(4x+8) - \ln(7x+8)}{x}$$
$$\overset{(L'H)}{=} \lim_{x \to 0} \left[\frac{4}{4x+8} - \frac{7}{7x+8}\right] = -\frac{3}{8}.$$

Ten Point Bonus: Prove

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

whenever x > 0. // Let $f(x) = \tan^{-1}(x) + \tan^{-1}(x^{-1})$ for x > 0. Then

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2} = 0$$

for x > 0. Consequently, f is continuous on $(0,\infty)$, and must be a constant function. Since $f(1) = 2 \cdot \tan^{-1}(1) = \pi/2$, $\tan^{-1}(x) + \tan^{-1}(x^{-1}) = \pi/2$ for x > 0.//

11. (15 pts.) Very carefully sketch the function

$$f(x) = 3x^4 - 4x^3 = 3x^3\left(x - \frac{4}{3}\right)$$

Do this only after you have completely analyzed its behavior.

Analysis: $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$ Clearly $f'(x) = 0$ if, and only if $x = 0$ or $x = 1$. The sign situation for the derivative is below.

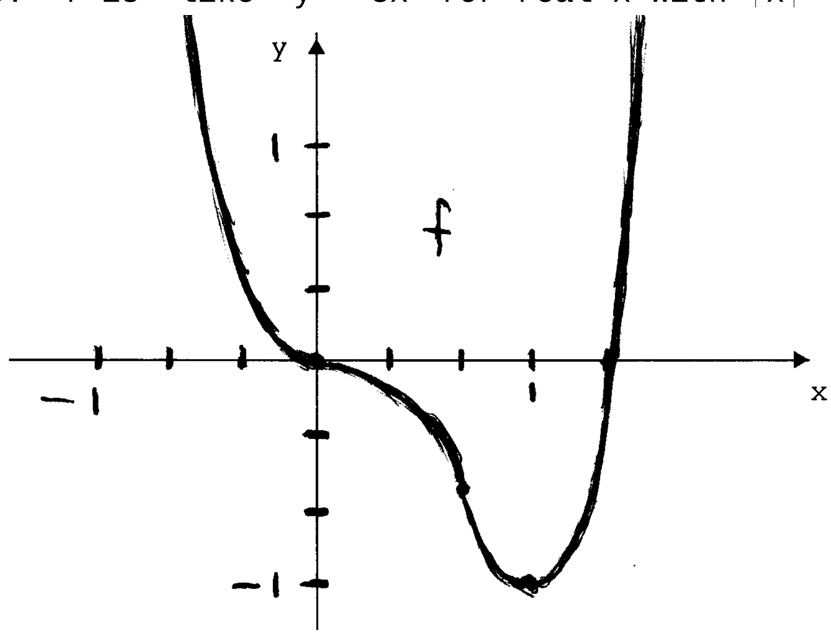
$f' < 0$		$f' < 0$		$f' > 0$
↘	0	↘	1	↗

$f''(x) = 36x^2 - 24x = 36x(x - (2/3))$ Clearly $f''(x) = 0$ if, and only if $x = 0$ or $x = 2/3$. The sign situation for the second derivative is below.

$f'' > 0$		$f'' < 0$		$f'' > 0$
$\cup$	0	$\cap$	2/3	$\cup$

Dots to Connect: $f(4/3) = f(0) = 0$, $f(1) = -1$, $f(2/3) = -16/27$

Additional info: f is "like" $y = 3x^4$ for real x with $|x|$ large.



12. (5 pts.)   A particle moves along the x-axis with the acceleration function $a(t) = 12t - 4$, initial position $x(0) = 0$, and initial velocity $v(0) = -10$. Find the particle's position function $x(t)$.//   By doing some routine antidifferentiations and using the initial conditions, you can obtain $v(t) = 6t^2 - 4t - 10$ so that $x(t) = 2t^3 - 2t^2 - 10t$.

Silly Ten Point Bonus: On Page 4 of 5.