**Read Me First:** *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

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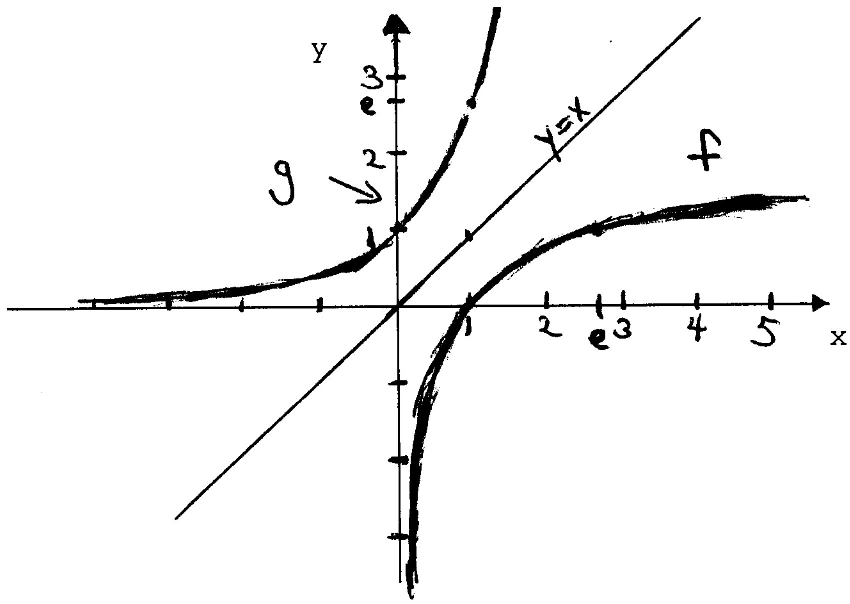
1. (15 pts.) (a) (5 pts.) Carefully sketch both f(x) = ln(x) and g(x) = $e^x$ on the coordinate system below. **Label very carefully**.




(b) (10 pts.) Evaluate each of the following limits.

$\lim_{x\to\infty} \ln(x) = \infty$ $\lim_{x\to 0^+} \ln(x) = -\infty$

$\lim_{x\to\infty} e^x = \infty$ $\lim_{x\to -\infty} e^x = 0$

$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

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2. (10 pts.) If $f(x) = \frac{1}{x}$ , find

$$\frac{f(x+h) - f(x)}{h},$$

and simplify as much as possible algebraically. Kindly observe that there are no limits of any sort being taken here.

$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \ldots = \frac{-1}{(x+h)(x)}$ *if* $h \neq 0$ after doing the obvious legal algebraic magic.

3. (25 pts.) For each of the following, find the limit if the limit exists. If the limit fails to exist, say so. Be as precise as possible here. [Work on the back of Page 1 of 4 if you run out of room here.]

(a) $\lim_{x \to 2} \frac{x^2-x-2}{x^2-1} = \frac{0}{3} = 0$

(b) $\lim_{x \to -1} \frac{x^2-x-2}{x^2-1} = \lim_{x \to -1} \frac{(x+1)(x-2)}{(x+1)(x-1)}$

$= \lim_{x \to -1} \frac{x-2}{x-1} = \frac{3}{2}$

(c) $\lim_{x \to 4} \frac{x^{1/2} - 2}{4 - x} = \ldots = \lim_{x \to 4} \frac{-1}{x^{1/2} + 2} = \frac{-1}{4}$

Here, of course, you do the obvious "rationalize the numerator" song and dance.

(d) $\lim_{x \to 4^-} \frac{2x - 8}{|x - 4|} = \lim_{x \to 4^-} \frac{2(x-4)}{-(x-4)} = -2$

(e) $\lim_{x \to 1} (20x^2 - 4)^{1/2} = (20(1)^2 - 4)^{1/2} = 4$

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Silly 10 point Bonus Problem: Let

$$f(x) = (\ln(x) - \ln(3))\cdot(e^x - e),\ x > 0.$$

Determine the open intervals where f(x) > 0 and the open intervals where f(x) < 0.

The key thing, that you might guess by considering the graphs, is that both the natural log and the exponential are "order preserving". [Later we shall say that they are "increasing."] This allows you to say where each of the factors is positive and where each is negative.

First,

$$\ln(x) - \ln(3) > 0 \Leftrightarrow \ln(x) > \ln(3) \Leftrightarrow x > 3$$

and

$$\ln(x) - \ln(3) < 0 \Leftrightarrow \ln(x) < \ln(3) \Leftrightarrow 0 < x < 3.$$

Next,

$$e^x - e > 0 \Leftrightarrow e^x > e^1 \Leftrightarrow x > 1$$

and

$$e^x - e < 0 \Leftrightarrow e^x < e^1 \Leftrightarrow x < 1.$$

Finally, by building a sign chart in our heads, we can see now that f(x) > 0 when x $\varepsilon$ (0,1) or x $\varepsilon$ $(3,\infty)$, and f(x) < 0 when x $\varepsilon$ (1,3).

4. (15 pts.) Suppose that

$$h(x) = \begin{cases} 2x - x^2 & , \textit{if } x > 1 \\ 2 & , \textit{if } x = 1 \\ \dfrac{4x^4 - 2}{3 + x^4} & , \textit{if } x < 1 \end{cases}$$

Evaluate each of the following easy limits.

(a) $\lim\limits_{x \to \infty} h(x) = \lim\limits_{x \to \infty} (2x - x^2) = -\infty$

(b) $\lim\limits_{x \to -\infty} h(x) = \lim\limits_{x \to -\infty} \dfrac{4x^4 - 2}{3 + x^4} = \lim\limits_{x \to -\infty} \dfrac{4 - \frac{2}{x^4}}{\frac{3}{x^4} + 1} = 4$

(c) $\lim\limits_{x \to 2} h(x) = \lim\limits_{x \to 2} (2x - x^2) = 0$

(d) $\lim\limits_{x \to 0} h(x) = \lim\limits_{x \to 0} \dfrac{4x^4 - 2}{3 + x^4} = \dfrac{-2}{3}$

(e) $\lim\limits_{x \to 1} h(x)$ doesn't exist since $\lim\limits_{x \to 1^-} h(x) = \lim\limits_{x \to 1^-} \dfrac{4x^4 - 2}{3 + x^4} = \dfrac{1}{2}$

and $\lim\limits_{x \to 1^+} h(x) = \lim\limits_{x \to 1^+} (2x - x^2) = 1$.

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5. (10 pts.) Here are five trivial limits to evaluate:

(a) $\lim\limits_{x \to -\infty} (-4) = -4$

(b) $\lim\limits_{h \to \infty} (-\pi h) = -\infty$

(c) $\lim\limits_{z \to -\infty} \dfrac{|\pi z|}{2z} = \lim\limits_{z \to -\infty} \dfrac{-\pi z}{2z} = \dfrac{-\pi}{2}$

(d) $\lim\limits_{x \to \infty} \left( \dfrac{\pi}{2} - \dfrac{3}{\ln(x)} \right) = \dfrac{\pi}{2}$

(e) $\lim\limits_{x \to -\infty} e^{-x^2} = 0$

6. (5 pts.) Using complete sentences and appropriate notation, provide the precise mathematical definition of a **function** f //

A function f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by f(x). [Note: The uniqueness of the output is critical.] *Definition 1.1.2.*

7. (5 pts.) Express the following function in piecewise defined form without using absolute values:

$$f(x) = |x-2| + |x|$$

$$= \begin{cases} 2-2x\ , & x < 0 \\ 2\ , & 0 \le x < 2 \\ 2x-2\ , & 2 \le x \end{cases}$$

after you carefully unwrap the absolute value functions.

8. (5 pts.) Express $f(x) = 2\cdot\cos^2(5x^4)$ as the composition of two functions g and h with $f = g\circ h$, that is find g and h so that $f(x) = (g\circ h)(x)$.

There are a multitude of appropriate g,h pairs. Here are a few:
$g(x) = 2x$ and $h(x) = \cos^2(5x^4)$; $g(x) = 2\cdot\cos^2(x)$ and $h(x) = 5x^4$;
$g(x) = x$ and $h(x) = 2\cdot\cos^2(5x^4)$; $g(x) = 2\cdot\cos^2(5x^4)$ and $h(x) = x$;
$g(x) = 2\cdot\cos^2(5x)$ and $h(x) = x^4$; $g(x) = 2x^2$ and $h(x) = \cos(5x^4)$;

Undoubtably, you can find a few more. Each "check" requires computing the composition with the given pair. Check??

9. (10 pts.) Evaluate each of the following thorny limits:

(a)

$$\lim_{x\to\infty} [(x^2+4x+4)^{1/2} - x] = \lim_{x\to\infty} \frac{4x + 4}{(x^2+4x+4)^{1/2} + x}$$

$$= \lim_{x\to\infty} \frac{4 + \dfrac{4}{x}}{\left(1+\dfrac{4}{x}+\dfrac{4}{x^2}\right)^{1/2} + 1}$$

$$= \frac{4}{2} = 2$$

by saying "Rationalize the numerator." Of course, if you were really paying attention algebraically, you may have handled this more easily by noting that

$$\lim_{x\to\infty} [(x^2+4x+4)^{1/2} - x] = \lim_{x\to\infty} (((x+2)^2)^{1/2} - x)$$

$$= \lim_{x\to\infty} (|x+2| - x) = \lim_{x\to\infty}(x+2-x) = 2.$$

(b) $\displaystyle\lim_{x\to 0} \frac{(9+x)^{1/2} - 3}{x} = \lim_{x\to 0} \frac{1}{(9+x)^{1/2} + 3} = \frac{1}{6}$ after muttering the usual "rationalize the numerator" incantation.

Silly 10 point Bonus Problem: Let

$$f(x) = (\ln(x) - \ln(3))\cdot(e^x - e),\ x > 0.$$

Determine the open intervals where $f(x) > 0$ and the open intervals where $f(x) < 0$. [Work on the back of page 3 of 4.] My answer is on Page 2 of 4 following Problem 3.