Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.*

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1. (25 pts.) Compute the derivatives of the following functions. You may use any of the rules of differentiation that are at your disposal. Do not attempt to simplify the algebra in your answers.

(a) $f(x) = 4x^{12} - 7x^{-6} + 8\tan(x)$ $f'(x) = 48x^{11} + 42x^{-7} + 8\sec^2(x)$

(b) $g(x) = (4x^{1/2} - 8x^{-1/4})\sec(x)$

$$g'(x) = (2x^{-1/2} + 2x^{-5/4})\sec(x) + (4x^{1/2} - 8x^{-1/4})\sec(x)\tan(x)$$

(c) $h(t) = \dfrac{5\cot(t)}{12t^{2/3}}$

$$h'(t) = \frac{-5csc^2(t)(12t^{2/3}) - 5\cot(t)(8t^{-1/3})}{(12t^{2/3})^2} = \frac{-60t^{2/3}csc^2(t) - 40t^{-1/3}\cot(t)}{144t^{4/3}}$$

(d) $y = \sin^3(\tan(2\theta + 1))$

$$\frac{dy}{d\theta} = 3\sin^2(\tan(2\theta+1))\cos(\tan(2\theta+1))\sec^2(2\theta+1)(2) = 6\sin^2(\tan(2\theta+1))\cos(\tan(2\theta+1))\sec^2(2\theta+1)$$

(e) $L(z) = \cos(8z^4) + 4csc(\frac{\pi}{2}) - 4csc(\frac{z}{2})$

$$\frac{dL}{dz}(z) = -\sin(8z^4)(32z^3) + 0 - (4)(-1)csc(\frac{z}{2})\cot(\frac{z}{2})(\frac{1}{2}) = -32z^3\sin(8z^4) + 2csc(\frac{z}{2})\cot(\frac{z}{2})$$

9(b) Using only the mathematical definition of limit, provide a complete proof that

$$\lim_{x \to -1} (7x + 5) = -2.$$

Proof: Let ε > 0 be arbitrary. Set δ = ε/7. Observe that δ > 0. Suppose now that x satisfies $0 < |x - (-1)| < \delta$. We now verify that $0 < |x - (-1)| < \delta$ implies $|(7x + 5) - (-2)| < \varepsilon$. Now

$$0 < |x - (-1)| < \delta \Rightarrow |x + 1| < \varepsilon/7 \Rightarrow 7|x + 1| < \varepsilon \Rightarrow |7x + 7| < \varepsilon \Rightarrow |(7x + 5) - (-2)| < \varepsilon.$$

Since, given an arbitrary ε > 0, we have produced a number δ > 0 such that, if x satisfies $0 < |x - (-1)| < \delta$, then $|(7x+5) - (-2)| < \varepsilon$, we have proved (7x + 5) —> -2 as x —> -1. [Without *scratching*.]

2. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definitions for **continuity** of a function f(x) at a point x = a.// A function f is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

(b) Is there a real number k, that will make the function f(x) defined below continuous at x = 0 ? Either find the value for k and using the definition, prove that it makes f continuous at x = 0, or explain why there cannot be such a number k. Suppose

$$f(x) = \begin{cases} \tan(\pi x)/(4x) & , x \neq 0 \\ k & , x = 0 \end{cases}$$

From the definition of continuity at a point, in order for f to be continuous at x = 0, it is necessary and sufficient for

$$k = f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\tan(\pi x)}{4x} = \frac{\pi}{4}.$$

The last equation in this string of equations may be obtained correctly in a couple of ways at this stage. Here's one such:

$$\lim_{x \to 0} \frac{\tan(\pi x)}{4x} = \lim_{x \to 0} \left(\frac{\pi}{4} \cdot \frac{\sin(\pi x)}{\pi x} \cdot \frac{1}{\cos(\pi x)} \right) = \frac{\pi}{4}.$$

The limits involving the second and third factors both equal 1. Another way is to reveal that the limit is a derivative!!

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3. (10 pts.) (a) Using complete sentences and appropriate notation, state the Intermediate Value Theorem.// If f is continuous on a closed interval [a,b], and k is any number between f(a) and f(b), inclusive, then there is a number x_0 in the interval [a,b] with $f(x_0) = k$.//
(b) Use the Intermediate Value Theorem to prove the equation

$$\sin(x) = -\frac{3}{4}$$

has at least one real solution in the interval [5π/2, 7π/2].
//Let $f(x) = \sin(x)$ on the closed interval [5π/2, 7π/2]. Since sine is continuous on the real line, f is continuous on the given closed interval. Now f(5π/2) = sin((π/2) + 2π) = sin(π/2) = 1 and f(7π/2) = sin((3π/2) + 2π) = sin(3π/2) = -1. The number k = -3/4 is between 1 = f(5π/2) and -1 = f(7π/2). Consequently, the Intermediate Value Theorem implies there is at least one number x_0 in [5π/2, 7π/2] where $f(x_0) = -3/4$ or $\sin(x_0) = -3/4$.

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4. (5 pts.) Reveal all the details in showing how to use the squeezing theorem to obtain the following limit:

$$\lim_{x \to 0} x^2 \sin^2\left(\frac{1}{x}\right) = 0.$$

Explanation: Observe that for $x \neq 0$, $-1 \leq \sin(1/x) \leq 1$ implies $0 \leq \sin^2(1/x) \leq 1$ and thus, that $0 \leq x^2\sin^2(1/x) \leq x^2$.
Since $x^2 \to 0$ as $x \to 0$, the squeezing theorem implies $x^2\sin^2(1/x) \to 0$ as $x \to 0$.

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5. (15 pts.) (a) (10 pts.) Using implicit differentiation, compute dy/dx and d^2y/dx^2 when $x^2 + y^3 = -9$. **Label your expressions correctly or else.**

First, pretend that y is a function of x. Then

$$x^2 + y^3 = -9 \quad \Rightarrow \quad \frac{d}{dx}(x^2 + y^3) = \frac{d}{dx}(-9)$$
$$\Rightarrow \quad 2x + 3y^2\frac{dy}{dx} = 0$$
$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{3y^2}.$$

Thus,

$$\frac{dy}{dx} = -\frac{2x}{3y^2} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{2x}{3y^2}\right)$$
$$= -\frac{2}{3}\frac{d}{dx}\left(\frac{x}{y^3}\right)$$
$$= -\frac{2}{3}\left(\frac{y^3 - x(3y^2)(dy/dx)}{y^6}\right)$$
$$= -\frac{2}{3}\left(\frac{y^3 + 2x^2}{y^6}\right).$$

(b) (5 pts.) Obtain an equation for the line tangent to the graph of $x^2 + y^3 = -9$ at the point $(-1,-(10)^{1/3})$.

$$y - (-(10)^{1/3}) = -\frac{2(-1)}{3((-1)(10^{1/3})^2}(x - (-1))$$

or

$$y + 10^{1/3} = \frac{2}{3(10)^{2/3}}(x + 1).$$

The slope above is the implicit derivative evaluated at the point in question, $(-1,-(10)^{1/3})$. It can also be obtained otherwise.

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6. (10 pts.) (a) Find formulas for Δy and the differential dy when $y = x^3$. Label your expressions correctly using an equal sign.

Evidentally $dy = 3x^2dx$ is cheap thrills, but Δy is slightly messier.

$$\Delta y = (x + \Delta x)^3 - x^3$$
$$= [\,x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3] - x^3$$
$$= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

(b) Use an appropriate linear approximation formula to estimate $(101)^{1/2}$.

Let $f(x) = x^{1/2}$ and set $x_0 = 100$. Then the local linear approximation at x_0 is given by

$$x^{1/2} \approx x_0^{1/2} + \frac{1}{2}x_0^{-1/2}(x - x_0) = 100^{1/2} + \frac{1}{2}\cdot\frac{1}{100^{1/2}}(x - 100)$$

for all x. When $x = 101$,

$$101^{1/2} \approx 100^{1/2} + \frac{1}{2}\cdot\frac{1}{100^{1/2}}(101 - 100) = 10 + \frac{1}{2}\cdot\frac{1}{10}\cdot 1 = 10 + \frac{1}{20} = 10.05.$$

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7. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative, f′(x), of a function f(x).// The function f′ defined by the equation

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}.$$

is called the derivative of f with respect to x. The domain of f′ consists of all x in the domain of f for which the limit above exists.//
(b) Using only the definition of the derivative as a limit, show all steps of the computation of f′(x) when f(x) = $4x^{-1}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to o}\frac{\left(\frac{4}{x+h} - \frac{4}{x}\right)}{h} \\ &= \lim_{h\to 0}\frac{-4h}{hx(x+h)} \\ &= \lim_{h\to 0}\frac{-4}{x(x+h)} = -\frac{4}{x^2} \quad \text{for } x \neq 0. \end{aligned}$$

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8. (5 pts.) Pretend f is a magical function that has the property that at x = 2π the tangent line f is actually defined by the equation y = -4π(x - π) + $3\pi^2$. Obtain

(a) f′(2π) = -4π (b) f(2π) = $-\pi^2$ //Conceptual points: (a) The slope of the tangent line at a point is the value of the derivative at the point. (b) The tangent line shares the point of tangency. So the y value of that point is the same as the function value at that point.

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9. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for

(*) $$\lim_{x\to a} f(x) = L.$$

Suppose that f is a function that is defined everywhere in some open interval containing x = a, except possibly at x = a. We write (*) if L is a number such that for each ε > 0 we can find a δ > 0, such that if x is in the domain of f and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.//
(b) Using only the mathematical definition of limit, provide a complete proof that

$$\lim_{x\to -1}(7x+5) = -2.$$

This may be found at the bottom of Page 1 of 4.

Silly 10 point Bonus Problem: Prove that if sine and cosine have a derivative at one point x_0, then sine and cosine have a derivative at every real number x and write sin′(x) and cos′(x) in terms of sin′(x_0) and cos′(x_0). [Say where your work is. It won’t fit here. **Hint:** Trig or treat derivative definition.]
An answer may be found in c1-t2-bo.pdf!!