Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.*

Silly 10 point Bonus Problem: Prove that if sine and cosine have a derivative at one point x_0, then sine and cosine have a derivative at every real number x and write sin′(x) and cos′(x) in terms of $\sin'(x_0)$ and $\cos'(x_0)$. [Say where your work is. It won't fit here. **Hint:** Trig or treat derivative definition.]

It turns out that to obtain this foolishness, we need only begin with the definition, mix in the magic number zero at the correct moment to introduce x_0 into the mix, and then utilize a couple of our old trigonometric identity friends to produce derivatives given by suitable limits.

First, pretend that $\sin'(x_0)$ and $\cos'(x_0)$ exist for some real number x_0. This means, of course, that both of the following limits exist as honest to goodness real numbers:

$$\lim_{h\to 0}\frac{\sin(x_0+h)-\sin(x_0)}{h}=\sin'(x_0)$$

and

$$\lim_{h\to 0}\frac{\cos(x_0+h)-\cos(x_0)}{h}=\cos'(x_0).$$

Now use the definition of the derivative with respect to sine.

$$\begin{aligned}\sin'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin((x-x_0)+(x_0+h))-\sin((x-x_0)+x_0)}{h}\end{aligned}$$

At this point we bring out our friendly sum of angles identity for sine to transmogrify the numerator. Since

$$\sin((x-x_0)+(x_0+h)) = \sin(x-x_0)\cos(x_0+h)+\cos(x-x_0)\sin(x_0+h)$$

and

$$\sin((x-x_0)+x_0) = \sin(x-x_0)\cos(x_0)+\cos(x-x_0)\sin(x_0),$$

by doing the obvious factoring out of the terms $\sin(x-x_0)$ and $\cos(x-x_0)$, we have

$$\begin{aligned}\sin'(x) &= \lim_{h\to 0}\frac{\sin((x-x_0)+(x_0+h))-\sin((x-x_0)+x_0)}{h}\\ &= \lim_{h\to 0}\left[\sin(x-x_0)\frac{\cos(x_0+h)-\cos(x_0)}{h}+\cos(x-x_0)\frac{\sin(x_0+h)-\sin(x_0)}{h}\right]\\ &= \sin(x-x_0)\cos'(x_0)+\cos(x-x_0)\sin'(x_0).\end{aligned}$$

Here, of course, we have invoked our hypothesis regarding the existence of derivatives.

The derivative of cosine involves similar chicanery with the key difference being in the trigonometric identity used. Instead of the sum of angles identity for sine, we use the corresponding identity for cosine.

Again we begin by using the definition of the derivative and use zero in an appropriate form.

$$\cos'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$
$$= \lim_{h\to 0} \frac{\cos((x-x_0)+(x_0+h)) - \cos((x-x_0)+x_0)}{h}$$

At this point we bring out our friendly sum of angles identity for cosine to change the form of the numerator. Since

$$\cos((x-x_0)+(x_0+h)) = \cos(x-x_0)\cos(x_0+h) - \sin(x-x_0)\sin(x_0+h)$$

and

$$\cos((x-x_0)+x_0) = \cos(x-x_0)\cos(x_0) + \sin(x-x_0)\sin(x_0),$$

by doing the obvious factoring out of the terms $\sin(x-x_0)$ and $\cos(x-x_0)$, we have

$$\cos'(x) = \lim_{h\to 0} \frac{\cos((x-x_0)+(x_0+h)) - \cos((x-x_0)+x_0)}{h}$$
$$= \lim_{h\to 0} \left[\cos(x-x_0)\frac{\cos(x_0+h)-\cos(x_0)}{h} - \sin(x-x_0)\frac{\sin(x_0+h)-\sin(x_0)}{h} \right]$$
$$= \cos(x-x_0)\cos'(x_0) - \sin(x-x_0)\sin'(x_0).$$

Here, again, we have invoked our hypothesis regarding the existence of derivatives.

Here, somewhat more briefly, are the two derivative formulae:

$$\sin'(x) = \sin(x-x_0)\cos'(x_0) + \cos(x-x_0)\sin'(x_0),$$

and

$$\cos'(x) = \cos(x-x_0)\cos'(x_0) - \sin(x-x_0)\sin'(x_0).$$

What was the motivation for this noise?? Obviously the usual computation of the derivatives of sine and cosine using the definition of the derivative is the culprit. Note that the usual argument is just the above magic where x_0 is conveniently equal to 0. What is necessary to complete things is to actually compute the derivatives of the two functions at x_0, and thus we can see now how the two "hard" limits for sine and cosine at zero arose, namely

$$\sin'(0) = \lim_{h\to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{h\to 0} \frac{\sin(h)}{h}$$

and

$$\cos'(0) = \lim_{h\to 0} \frac{\cos(0+h) - \cos(0)}{h} = \lim_{h\to 0} \frac{\cos(h) - 1}{h}.$$