Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.*

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1. (10 pts.) Determine the maximum and minimum values of the function $f(x) = x^3 - 2x^2$ on the interval [1,4] and where they occur. Before you attempt to find them, explain how you know f(x) actually has both a maximum and a minimum. [Hint: Pay attention to the domain. Watch out for "critical points" that aren't.]// First, the polynomial function $f(x) = x^3 - 2x^2$ is continuous on the closed interval [1,4]. Consequently, the Extreme-Value Theorem implies that f has both a maximum and a minimum on the interval. We expect to find them by considering the function values of f at critical points and end points. Now $f'(x) = 3x^2 - 4x = 3x(x - (4/3))$. Thus, f has exactly one critical point at x = 4/3. Since f(1) = -1, f(4/3) = -32/27, and f(4) = 32, the maximum value is 32 and occurs at x = 4, and the minimum value is -32/27 and occurs at x = 4/3.

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2. (10 pts.) (a) Use logarithmic differentiation to find dy/dx when

$$y = (x^4 + 1)^{\sec^{-1}(x)}.$$

$$y = (x^4+1)^{\sec^{-1}(x)} \Rightarrow \ln(y) = \sec^{-1}(x)\ln(x^4+1)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{|x|(x^2-1)^{1/2}}\ln(x^4+1) + \sec^{-1}(x)\frac{4x^3}{x^4+1}.$$

Thus,

$$\frac{dy}{dx} = \left[\frac{1}{|x|(x^2-1)^{1/2}}\ln(x^4+1) + \sec^{-1}(x)\frac{4x^3}{x^4+1}\right](x^4+1)^{\sec^{-1}(x)}.$$

(b) Find dy/dx by using implicit differentiation when

$$\tan^{-1}(xy) = \tan(x + y).$$

$$\tan^{-1}(xy) = \tan(x+y) \Rightarrow \frac{d}{dx}\tan^{-1}(xy) = \frac{d}{dx}\tan(x+y)$$

$$\Rightarrow \frac{1}{1+(xy)^2}\left(y + x\frac{dy}{dx}\right) = \sec^2(x+y)\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \left[\frac{x}{1+(xy)^2} - \sec^2(x+y)\right]\frac{dy}{dx} = \sec^2(x+y) - \frac{y}{1+(xy)^2}.$$

Thus,

$$\frac{dy}{dx} = \frac{\left[\sec^2(x+y) - \frac{y}{1+(xy)^2}\right]}{\left[\frac{x}{1+(xy)^2} - \sec^2(x+y)\right]}.$$

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3. (10 pts.) (a) State the Mean Value Theorem of Differential Calculus. Use a complete sentence and appropriate notation.
// If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there is at least one number c in the interval (a,b) with

$$f(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) By using the Mean Value Theorem on the interval [3,4] with respect to the function $f(x) = x^{1/2}$, show that $1.71 < 3^{1/2} < 1.75$.
// By applying the M.V.T. to the function above on the interval [3,4], evidently there is a number c with $3 < c < 4$ so that

$$\frac{1}{2(c)^{1/2}} = \frac{4^{1/2} - 3^{1/2}}{4-3} = 2 - 3^{1/2}.$$

This allows us to write $3^{1/2}$ exactly as

$$3^{1/2} = 2 - \frac{1}{2c^{1/2}}$$

Since the function $g(x) = 1/(2x^{1/2})$ is decreasing on the interval [3.4], it follows that

$$2 - \frac{1}{2(3)^{1/2}} < 3^{1/2} < 2 - \frac{1}{2(4)^{1/2}}.$$

Clearly the right-hand inequality above is just $3^{1/2} < 1.75$. Now

$$2 - \frac{1}{2(3)^{1/2}} < 3^{1/2} \;\Rightarrow\; 2 < \frac{7}{6}(3)^{1/2} \;\Rightarrow\; \frac{12}{7} < 3^{1/2}.$$

Since $1.71 < 12/7$, we are done.

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4. (10 pts.) Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin^{-1}(3x)}{x} = 3$

by using L'Hopital's Rule routinely. Along the way you would run into the function g below and its derivative. You don't actually have to use L'Hopital's Rule if you recognized that

$$\lim_{x\to 0} \frac{\sin^{-1}(3x)}{x} = \lim_{x\to 0} \frac{\sin^{-1}(3x) - \sin^{-1}(0)}{x}$$
$$= g'(0) = 3$$

when $g(t) = \sin^{-1}(3t)$ so that $g'(t) = \dfrac{3}{(1-(3t)^2)^{1/2}}$.

(b) $\displaystyle\lim_{x\to +\infty} (1 - (3/x))^x = \lim_{x\to +\infty} e^{x\ln(1-3x^{-1})} = e^{-3}$ since

$$\lim_{x\to +\infty} x\ln(1-3x^{-1}) = \lim_{x\to +\infty} \frac{\ln(1-3x^{-1})}{x^{-1}}$$

$(L'H + algebra)$

$$= \lim_{x\to +\infty} \frac{-3}{1-\frac{3}{x}} = -3.$$

5. (10 pts.) A rectangular area of 1600 square feet is to be fenced. Three of the sides will use fencing costing \$4.00 per running foot, and the remaining side will use fencing costing \$1.00 per running foot. Find the dimensions of the rectangle which lead to the least cost to fence the area, and prove that these dimensions actually result in the minimum cost by means of an appropriate analysis.
// Let the two perpendicular side lengths be denoted by "x" and "y". Then suppose two of the "x" sides and one of the "y" sides cost \$4 per foot, and the remaining "y" side costs \$1 per foot. Then the cost in dollars, in terms of the lengths of the sides, is given by $C = 4(2x) + 4y + y = 8x + 5y$. Since the area is the constant 1600, we can use the equation $xy = 1600$ to transform the cost into a function of x alone. [You could also turn the cost info a function of y.] Using this information and that the side length must be positive, the function we want to minimize is

$$C(x) = 8x + 8000x^{-1}$$

for $x > 0$. Since $C'(x) = (8/x^2)(x - (1000)^{1/2})(x + (1000)^{1/2})$, C has exactly one critical point at $x = (1000)^{1/2} = 10(10)^{1/2}$ on the interval $(0,\infty)$. Plainly $0 < x < 1000)^{1/2}$ implies $C'(x) < 0$, and $x > 1000)^{1/2}$ implies $C'(x) > 0$. Thus, the cost function has an absolute minimum at $x = (1000)^{1/2}$. The other dimension needed is $y = 1600/(1000)^{1/2} = 16(10)^{1/2}$. Both numbers represent *feet*.

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6. (5 pts.) Rolle's Theorem states that if f(x) is continuous on [a,b] with f(a) = f(b) = 0 and differentiable on (a,b), then there is a number c in (a,b) such that f′(c) = 0. **Find an example of a function f(x) defined on [-1,1] with f being differentiable on (-1,1) with f(-1) = f(1) = 0 but such that there is no number c in (-1,1) with f′(c) = 0.** [Hint: Which hypothesis above must you violate??]
// Obviously the example we cook up must fail to be *continuous* on [-1,1], and must be differentiable on (-1,1). Let

$$f(x) = \begin{cases} 0, & x = -1 \text{ or } x = 1 \\ x, & -1 < x < 1 \end{cases}$$

Then $f'(x) = 1$ for $-1 < x < 1$. Consequently, $f'(c) \neq 0$ for every c with $-1 < c < 1$. So the conclusion of Rolle's Theorem is false.

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7. (5 pts.) Find the function f(x) which satisfies the following two equations: $f'(x) = 43x^{42} + 5x^4$ for all x and $f(-1) = 4$.// First,

$$f'(x) = 43x^{42} + 5x^4 \;\Rightarrow\; f(x) = \int 43x^{42} + 5x^4\,dx$$
$$\Rightarrow\; f(x) = x^{43} + x^5 + C$$

for some number C. Then

$$4 = f(-1) \;\Rightarrow\; 4 = (-1)^{43} + (-1)^5 + C$$
$$\Rightarrow\; C = 6.$$

Consequently,

$$f(x) = x^{43} + x^5 + 6.$$

8. (5 pts.) Do you really understand equations with integral signs?? Find the function $g(x)$ that satisfies the following equation

$$\int g(x)\,dx = \sin(x) - x\cos(x) + C$$

From the definition of the word *antiderivative*,

$$g(x) = \frac{d}{dx}[\sin(x) - x\cos(x) + C]$$

$$= \sin'(x) - (x\cos(x))'$$

$$= \cos(x) - (\cos(x) - x\sin(x)) = x\sin(x).$$

9. (15 pts.) Evaluate each of the following integrals or anti-derivatives.

(a) $\displaystyle\int 3x^2 + \frac{1}{x} + 3\sec^2(x)\,dx = x^3 + \ln|x| + 3\tan(x) + C.$

(b) $\displaystyle\int \frac{2x^3 + 4x^2 + 1}{x^2}\,dx = \int 2x + 4 + x^{-2}\,dx = x^2 + 4x - x^{-1} + C.$

(c) $\displaystyle\int \frac{1}{1 + 4x^2} + (4x + 2)e^{x^2 + x}\,dx = \frac{1}{2}\tan^{-1}(2x) + 2e^{x^2+x} + C$

by using two obvious u-substitutions.

Silly 10 point Bonus Problem: Show how to use the Mean Value Theorem to prove the following result: If f is continuous on a closed interval [a,b] with $f'(x) < 0$ for each x in (a,b), then f is decreasing on the closed interval [a,b].

To prove that f is decreasing on [a,b], we must show that if x_1 and x_2 are two arbitrary numbers with

(*) $\qquad a \le x_1 < x_2 \le b,$

then under the hypotheses of the proposition, it follows that

(**) $\qquad f(x_1) > f(x_2).$

To this end, pretend that x_1 and x_2 are two numbers satisfying the inequality (*) above. It is plain that the function f satisfies the hypotheses of the Mean Value Theorem on the closed interval [a,b], and thus, also on the closed interval $[x_1,x_2]$. It follows from this that there is a number c in the open interval (x_1,x_2) where $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) < 0$ from the hypothesis, we have $f(x_2) - f(x_1) < 0$, which in turn implies that $f(x_2) < f(x_1)$. This, however, is equivalent to (**), and thus we are finished.//

10. (20 pts.) Fill in the blanks of the following analysis with the correct terminology. Then sketch the graph of the function analyzed on the coordinate system provided. Label carefully.

Let $f(x) = 2x^3 - 6x^2 + 4$. Then $f'(x) = 6x^2 - 12x$

$= 6(x - 0)(x - 2)$. Consequently, $x = 0$ and $x = 2$ are

critical (stationary) points of f. Since $f'(x) > 0$ for $x < 0$ or

$x > 2$, f is increasing on the set $(-\infty,0)\cup(2,\infty)$. Also,

because $f'(x) < 0$ when $0 < x < 2$, f is decreasing

on the interval $(0,2)$. Using the first derivative test, it

follows that f has a(n) relative (or local) maximum at

$x = 0$, and a(n) relative (or local) minimum at $x = 2$.

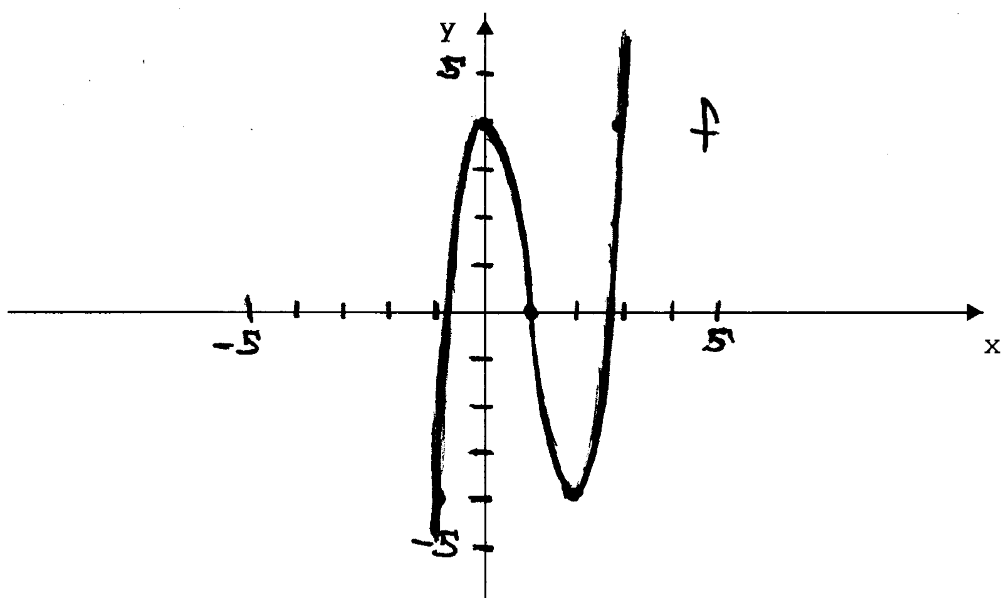
Since $f''(x) = 12(x - 1)$, we have $f''(1) = 0$, $f''(x) < 0$ when $x < 1$,

and $f''(x) > 0$ when $x > 1$. Thus, f is concave down

on the interval $(-\infty,1)$, f is concave up on $(1,\infty)$,

and f has a(n) inflection point at $x = 1$.

"Dots to Connect" : $f(0) = 4$, $f(1) = 0$, $f(2) = -4$ Besides $f(1) = 0$, f also has zeros at $x = 1 \pm 3^{1/2}$ since long division reveals that $f(x) = (x-1)(2x^2 - 4x -4)$. Fake using $f(-1) = -4$ and $f(3) = 4$.



Silly 10 point Bonus Problem: Show how to use the Mean Value Theorem to prove the following result: If f is continuous on a closed interval [a,b] with $f'(x) < 0$ for each x in (a,b), then f is decreasing on the closed interval [a,b]. [Say where your work is, for it won't fit here!] To be found on Page 4 of 5 ... .