Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.*

1. (25 pts.) Compute the derivatives of the following functions. You may use any of the rules of differentiation that are at your disposal. Do not attempt to simplify the algebra in your answers.

(a) $f(x) = 4x^6 - 7x^{-12} + 8\cos(x)$

$f'(x) = 24x^5 + 84x^{-13} - 8\sin(x)$

(b) $g(x) = (4x^4 - 8x^{-2})\sin(x)$

$g'(x) = (16x^3 + 16x^{-3})\sin(x) + (4x^4 - 8x^{-2})\cos(x)$

(c) $h(t) = \dfrac{\sec(t)}{10t^5}$

$$h'(t) = \frac{\sec(t)\tan(t)(10t^5) - \sec(t)(50t^4)}{100t^{10}}$$

(d) $y = \tan(csc(2\theta + 1))$

$$\frac{dy}{d\theta} = \sec^2(csc(2\theta+1))(-1)csc(2\theta+1)\cot(2\theta+1)(2)$$
$$= -2\sec^2(csc(2\theta+1))csc(2\theta+1)\cot(2\theta+1)$$

(e) $L(z) = \cot(4z^8) + 4\sec(\dfrac{\pi}{4}) - 4\sec(\dfrac{z}{2})$

$$\frac{dL}{dz}(z) = -32z^7 csc^2(4z^8) - 2\sec(\frac{z}{2})\tan(\frac{z}{2})$$

Silly 10 point Bonus Problem: Suppose that *x is measured in degrees*. With this hypothesis, compute $\sin'(x)$ and $\cos'(x)$.

The key here is to relate --- and distinguish between --- functions with degree arguments and functions with radian arguments since we know how to differentiate the latter.

To keep track of who is who, let $\sin_r(t)$ and $\cos_r(t)$ denote your friendly real sine and cosine functions whose argument t is measured in *radians* and let us denote the sine and cosine functions whose arguments are in *degree measure* by $\sin°(x)$ and $\cos°(x)$.

How are these varmints related?? A little thought reveals that $\sin°(x) = \sin_r(\pi x/180)$ and that $\cos°(x) = \cos(\pi x/180)$ for all x.

Since we know the derivatives of our friendly real number/radian measure versions of sine and cosine, by using chain rule and the relationship between the two flavors of functions, we have

$$(\sin°)'(x) = (\sin_r(\pi x/180))' = (\pi/180)\cos_r(\pi x/180) = (\pi/180)\cos°(x)$$

and

$$(\cos°)'(x) = (\cos_r(\pi x/180))' = -(\pi/180)\sin_r(\pi x/180) = -(\pi/180)\sin°(x).$$

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2. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition of **continuity** of a function f(x) at a point x = a.

// A function f is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

(b) Is there a nonzero real number k, that will make the function f(x) defined below continuous at x = 0 ? Either find the value for k and using the definition, prove that it makes f continuous at x = 0, or explain why there cannot be such a number k. Suppose

$$f(x) = \begin{cases} \tan(kx)/x \ , & x < 0 \\ 3x + 2k^2 \ , & x \geq 0 \end{cases}$$

From the definition of continuity at a point, in order for f to be continuous at $x = 0$, it is necessary and sufficient for f to be both left and right continuous. Since

$$2k^2 = f(0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3x + 2k^2)$$

and

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\tan(kx)}{x} = k \ ,$$

f is continuous at $x = 0$ if, and only if $2k^2 = k$. The only nonzero solution to this equation is k = 1/2. The last equation in this string of equations may be obtained correctly in a couple of ways at this stage. Here's one such:

$$\lim_{x \to 0^-} \frac{\tan(kx)}{x} = \lim_{x \to 0^-} \left(\frac{k}{1} \cdot \frac{\sin(kx)}{kx} \cdot \frac{1}{\cos(kx)} \right) = k.$$

The limits involving the second and third factors both equal 1.

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3. (10 pts.) (a) Using complete sentences and appropriate notation, state the Intermediate Value Theorem.// If f is continuous on a closed interval [a,b], and k is any number between f(a) and f(b), inclusive, then there is a number x_0 in the interval [a,b] with $f(x_0) = k$.//

(b) Use the Intermediate Value Theorem to prove the equation

$$x^4 + x - 1 = 0$$

has at least one real solution in the interval [-1, 1].

//Let $f(x) = x^4 + x - 1$ on the closed interval [-1, 1]. Since f is a polynomial, it is continuous on the given closed interval. Now f(1) = 1 and f(-1) = -1. The number k = 0 is between 1 = f(1) and -1 = f(-1). Consequently, the Intermediate Value Theorem implies there is at least one number x_0 in [-1, 1] where $f(x_0) = 0$ or $(x_0)^4 + (x_0) - 1 = 0$.//

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4. (5 pts.) Reveal all the details in showing how to use the squeezing theorem to obtain the following limit:

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

[Warning: A numerical value without an explanation is worth nothing.]

Explanation: Observe that for $x \neq 0$, $-1 \leq \sin(1/x) \leq 1$. Thus $x > 0$ implies that $-x \leq (x)\sin(1/x) \leq x$, and $x < 0$ implies that we have $x \leq (x)\sin(1/x) \leq -x$. Thus, if $x \neq 0$, then $-|x| \leq (x)\sin(1/x) \leq |x|$. Since $|x| \to 0$ as $x \to 0$, the squeezing theorem implies $(x)\sin(1/x) \to 0$ as $x \to 0$.

5. (5 pts.) Compute f″(x) when

$$f(x) = \sin(x^4).$$

$f'(x) = \cos(x^4)(4x^3) = 4x^3\cos(x^4).$

$f''(x) = 12x^2\cos(x^4) - 4x^3\sin(x^4)(4x^3) = 12x^2\cos(x^4) - 16x^6\sin(x^4).$

6. (5 pts.) Find an equation for the tangent line to the graph of

$$y = x\cos(3x)$$

when $x = \pi$. Since $y(x) = x\cos(3x)$ implies that

$y'(x) = \cos(3x) - 3x\sin(3x)$, we have $y(\pi) = \pi\cos(3\pi) = -\pi$ and

$y'(\pi) = \cos(3\pi) - 3\pi\sin(3\pi) = -1$. Tangent line equation:

$y+\pi = -1(x-\pi)$ *or* $y = -x$.

7. (10 pts.) (a) Find formulas for Δy and the differential dy when y = $x^2 - 2x + 1$. Label your expressions correctly.

Evidently $dy = (2x - 2)dx$ is cheap thrills, but Δy is slightly messier.

$$\begin{aligned}\Delta y &= ((x+\Delta x)^2-2(x+\Delta x)+1) - (x^2-2x+1)\\ &= [\,x^2+2x\Delta x+(\Delta x)^2-2x-2(\Delta x)+1\,] - [\,x^2-2x+1\,]\\ &= (2x-2)\Delta x + (\Delta x)^2.\end{aligned}$$

(b) Use an appropriate local linear approximation formula to estimate $(65)^{1/2}$.

Let $f(x) = x^{1/2}$ and set $x_0 = 64$. Then the local linear approximation at x_0 is given by

$$x^{1/2} \approx x_0^{1/2} + \frac{1}{2}x_0^{-1/2}(x-x_0) = 64^{1/2} + \frac{1}{2}\cdot\frac{1}{64^{1/2}}(x - 64)$$

for all x. When $x = 65$,

$$65^{1/2} \approx 64^{1/2}+\frac{1}{2}\cdot\frac{1}{64^{1/2}}(65-64) = 8+\frac{1}{2}\cdot\frac{1}{8}\cdot 1 = 8+\frac{1}{16} = \frac{129}{16} = 8.0625.$$

8. (5 pts.) The side of the square is measured to be 10 ft, with a possible error of ±0.1 ft. Estimate the percent error in the calculated area.

The area of a square is given by $A(x) = x^2$, where x is the length of a side. Using differentials, we may approximate the relative error as follows:

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2x_0dx}{x_0^2} = \frac{2\Delta x}{x_0} = \frac{(2)(\pm 0.1)}{10} = \pm 0.02 = \pm 2\% .$$

9. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative, $f'(x)$, of a function $f(x)$.
// The function f′ defined by the equation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

is called the derivative of f with respect to x. The domain of f′ consists of all x in the domain of f for which the limit above exists.//

(b) Using only the definition of the derivative as a limit, show all steps of the computation of $f'(x)$ when $f(x) = 4x^2 + 3$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to o} \frac{(4(x+h)^2+3) - (4x^2+3)}{h} \\ &= \lim_{h \to 0} \frac{8xh+4h^2}{h} \\ &= \lim_{h \to 0} (8x+4h) = 8x \quad \textit{for every real number } x. \end{aligned}$$

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10. (5 pts.) Pretend f is a magical function that has the property that at x = 2 the tangent line f is actually defined by the equation y = -4(x - 1) + 3. Obtain

(a) $f'(2) = -4$ (b) $f(2) = -1$ //Conceptual points: (a) The slope of the tangent line at a point is the value of the derivative at the point. (b) The tangent line shares the point of tangency. So the y value of that point is the same as the function value at that point.

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11. (10 pts.) A softball diamond is a square whose sides are 60 feet long. Suppose a player running from first to second base has a speed of 25 feet/second at the instant when she is 10 feet from second base. At what rate is the player's distance from home plate changing at that instant?

Let $x(t)$ denote the distance from first base to the runner, and let $y(t)$ denote the distance from home plate to the runner. Then from the Pythagorean Theorem,

$$(*) \qquad y^2(t) = 60^2 + x^2(t)$$

while the runner is between first and second base. Consequently, during this time interval, by differentiating with respect to t, we see

$$2y(t)y'(t) = 2x(t)x'(t) \; .$$

At the moment in question, t_0, $x(t_0) = 50$ and $x'(t_0) = 25$. Using (*) and doing a bit of algebra reveals that

$$y'(t_0) = \frac{x(t_0)x'(t_0)}{y(t_0)} = \frac{(50)(25)}{(10)(61)^{1/2}} = \frac{125}{(61)^{1/2}} \; ft./\text{sec.}$$

Silly 10 point Bonus Problem: Suppose that x *is measured in degrees*. With this hypothesis, compute $\sin'(x)$ and $\cos'(x)$. Say where your work is, for it won't fit here. **[See Page 1 of 4.]**