Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.*

1. (10 pts.) Determine the maximum and minimum values of the function

$$f(x) = 10x^2 - x^4$$

on the interval [-1,5] and where they occur. Before you attempt to find them, explain how you know f(x) actually has both a maximum and a minimum.
// First, the polynomial function $f(x) = 10x^2 - x^4$ is continuous on the closed interval [-1,5]. Consequently, the Extreme-Value Theorem implies that f has both a maximum and a minimum on the interval. We expect to find them by considering the function values of f at critical points and end points. Now

$$f'(x) = 20x - 4x^3 = -4x(x - \sqrt{5})(x + \sqrt{5}).$$

It follows that f has critical points at $x = 0$ and at $x = 5^{1/2}$ in (-1,5). Since f(-1) = 9, f(0) = 0, $f(5^{1/2}) = 25$, and f(5) = -375, the maximum value is 25 and occurs at $x = 5^{1/2}$, and the minimum is -375 and occurs at $x = 5$.

2. (10 pts.) (a) Use logarithmic differentiation to find dy/dx when

$$y = (x^3 - 2x)^{\ln(x)}.$$

$$y = (x^3-2x)^{\ln(x)} \Rightarrow \ln(y) = \ln(x)\ln(x^3-2x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(x^3-2x) + \ln(x)\frac{3x^2-2}{x^3-2x}.$$

Thus,

$$\frac{dy}{dx} = \left[\frac{1}{x}\ln(x^3-2x) + \ln(x)\frac{3x^2-2}{x^3-2x}\right](x^3-2x)^{\ln(x)}.$$

HOMEWORK: Section 4.3 #27

(b) Find dy/dx by using implicit differentiation when

$$x^3 + x\tan^{-1}(y) = e^y.$$

$$x^3 + x\tan^{-1}(y) = e^y \Rightarrow \frac{d}{dx}(x^3 + x\tan^{-1}(y)) = \frac{d}{dx}(e^y)$$
$$\Rightarrow 3x^2 + \tan^{-1}(y) + \frac{x}{1+y^2}\frac{dy}{dx} = e^y\frac{dy}{dx}$$
$$\Rightarrow \left[e^y - \frac{x}{1+y^2}\right]\frac{dy}{dx} = 3x^2 + \tan^{-1}(y).$$

Thus,

$$\frac{dy}{dx} = \frac{\left[3x^2 + \tan^{-1}(y)\right]}{\left[e^y - \frac{x}{1+y^2}\right]}.$$

HOMEWORK: Section 4.3 #51

3. (10 pts.) (a) State the Mean Value Theorem of Differential Calculus. Use a complete sentence and appropriate notation.

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there is at least one number c in the interval (a,b) with

$$f(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) Let $f(x) = x^{2/3}$, $a = -1$ and $b = 8$. Show there is no point c in the open interval (a,b) such that

(*) $$f'(c) = \frac{f(b) - f(a)}{b - a},$$

and explain why this does not violate the Mean-Value Theorem.

First, the reason this example does not violate the Mean-Value Theorem is that it does not satisfy all of the hypotheses. The function f is not differentiable at every point of $(-1,8)$ since it is not differentiable at $x = 0$.

Now observe that, for the given data, equation (*) is equivalent to

(**) $$c^{-1/3} = \tfrac{1}{2}$$

after a little routine algebraic magic is performed. If (**) is true for some real number c, then it follows that we must have $c = 8$, but 8 is in neither the interval $(-1,0)$ nor the interval $(1,8)$, where f is differentiable. Thus, (*) above fails to be true for every real number c in the interval $(a,b) = (-1,8)$.

HOMEWORK: Section 5.7 #16

4. (10 pts.) Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible.

(a) $$\lim_{x\to\pi/2^-} \sec(3x)\cos(5x) = \lim_{x\to\pi/2^-} \frac{\cos(5x)}{\cos(3x)} \overset{(L'H)}{=} \lim_{x\to\pi/2^-} \frac{5\sin(5x)}{3\sin(3x)} = -\frac{5}{3}.$$

HOMEWORK: Section 4.4 #23

(b) $$\lim_{x\to+\infty} \left[x - \ln(1 + 2e^x)\right] = \lim_{x\to+\infty} \ln\left(\frac{e^x}{1 + 2e^x}\right) = \lim_{x\to+\infty} \ln\left(\frac{1}{\frac{1}{e^x} + 2}\right) = \ln\left(\frac{1}{2}\right).$$

You may also use L'Hopital to deal with the argument of the natural log.

HOMEWORK: Section 4.4 #46

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5. (10 pts.) A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container with the least cost.

Let "x" denote the length of one side of the base, and let "y" denote the height of the container with both measured in centimeters. Then we must have $x^2y = 2000$, and if $C > 0$ is the cost of a square centimeter of the material used on the sides, the total cost of the materials, in terms of the dimensions is given by $T = (2C)(2x^2) + C(4xy) = 4C[x^2 + xy]$. Using the first equation, we may write T as a function of x alone as

$$T(x) = 4C[x^2 + 2000x^{-1}]$$

for $x > 0$. Since

$$T'(x) = 4C[2x - 2000x^{-2}] = \frac{8C(x-10)(x^2+10x+100)}{x^2}$$

for $x > 0$, it follows that $T' < 0$ for $0 < x < 10$ and $T' > 0$ for $x > 10$. Thus, $x = 10$ provides the minimum total cost. The dimensions are $x = 10$ centimeters and $y = 2000/10^2 = 20$ centimeters.//

HOMEWORK: Section 5.5 #21

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6. (5 pts.) Do you really understand equations with integral signs?? Find the function $g(x)$ that satisfies the following equation

$$\int g(x)\,dx = \sec^2(x) - \cos(x) + C$$

From the definition of the word *antiderivative*,

$$g(x) = \frac{d}{dx}[\sec^2(x) - \cos(x) + C]$$

$$= (\sec^2(x))' - (\cos(x))'$$

$$= 2\sec(x)[\sec(x)\tan(x)] - (-\sin(x)) = 2\sec^2(x)\tan(x) + \sin(x).$$

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7. (5 pts.) Solve the following initial-value problem:

$$\frac{dy}{dx} = \frac{x+1}{\sqrt{x}}, \quad y(1) = 0.$$

First,

$$\frac{dy}{dx} = \frac{x+1}{\sqrt{x}} \Rightarrow y(x) = \int \frac{x+1}{\sqrt{x}}\,dx = \int x^{1/2} + x^{-1/2}\,dx$$

$$\Rightarrow y(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

for some number C. Then

$$0 = y(1) \Rightarrow 0 = \frac{2}{3}(1)^{3/2} + 2(1)^{1/2} + C \Rightarrow C = -\frac{8}{3}.$$

Consequently,

$$y(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} - \frac{8}{3} \text{ for } x > 0.$$

Why $x > 0$?? **HOMEWORK: Section 6.2 #41(c)**

8. (8 pts.) Assume f is continuous everywhere. If

$$f'(x) = x^4(e^x - 3),$$

find all the critical points of f and at each stationary point apply the second derivative test to determine relative extrema, if possible. If the second derivative test fails at a critical point, apply the first derivative test to determine the true state of affairs there.

We may read off of the derivative that the critical points are $x = 0$ and $x = \ln(3)$. Since

$$f''(x) = 4x^3(e^x - 3) + x^4 e^x,$$

$f''(0) = 0$ and $f''(\ln(3)) = \ln^4(3)(3) > 0$. Thus, the second derivative test provides no information at $x = 0$, and implies that f has a relative minimum at $x = \ln(3)$.

When $x < \ln(3)$, $e^x < e^{\ln(3)}$, which implies $e^x - 3 < 0$. Thus, $f'(x) < 0$ when $x < \ln(3)$ and $x \neq 0$. The first derivative test now implies that f has neither a local max at $x = 0$ nor a local min there.//

9. (12 pts.) Evaluate each of the following integrals or anti-derivatives.

(a) $\displaystyle\int 3x^2 + \frac{1}{x^2+1} + 3csc^2(x)\,dx = x^3 + \tan^{-1}(x) - 3\cot(x) + C$

(b) $\displaystyle\int \frac{x^3 + x + 1}{x^2}\,dx = \int x + \frac{1}{x} + \frac{1}{x^2}\,dx = \tfrac{1}{2}x^2 + \ln|x| - x^{-1} + C$

(c) $\displaystyle\int \frac{1}{(1 - 4x^2)^{1/2}} + (4x - 2)e^{x^2 - x}\,dx = \tfrac{1}{2}\sin^{-1}(2x) + 2e^{x^2 - x} + C$

by using two obvious u-substitutions.

Silly 10 point Bonus Problem: Show how to use the Mean Value Theorem to prove the following result: If f is continuous on a closed interval [a,b] with $f'(x) > 0$ for each x in (a,b), then f is increasing on the closed interval [a,b].

To prove that f is increasing on [a,b], we must show that if x_1 and x_2 are two arbitrary numbers with

(*) $$a \le x_1 < x_2 \le b,$$

then under the hypotheses of the proposition, it follows that

(**) $$f(x_1) < f(x_2).$$

To this end, pretend that x_1 and x_2 are two numbers satisfying the inequality (*) above. It is plain that the function f satisfies the hypotheses of the Mean Value Theorem on the closed interval [a,b], and thus, also on the closed interval $[x_1,x_2]$. It follows from this that there is a number c in the open interval (x_1,x_2) where $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) > 0$ from the hypothesis, we have $f(x_2) - f(x_1) > 0$, which in turn implies that $f(x_2) > f(x_1)$. This, however, is equivalent to (**), and thus we are finished.//

10. (20 pts.) When f is defined by

$$f(x) = \frac{2x}{x^2+1}$$

on the real line, it turns out that

$$f'(x) = \frac{-2(x-1)(x+1)}{(x^2+1)^2} \quad \text{and} \quad f''(x) = \frac{4x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}.$$

(a) (3 pts.) What are the critical point(s) of f and what is the value of f at each critical point?

The critical points of f *are* $x_1 = -1$ *and* $x_2 = 1$. $f(-1) = -1$ *and* $f(1) = 1$.

(b) (3 pts.) Determine the open intervals where f is increasing or decreasing.

f *is decreasing on* $(-\infty,-1)\cup(1,\infty)$ *and increasing on* $(-1,1)$.

(c) (3 pts.) Determine the open intervals where f is concave up or concave down.

f *is concave down on* $(-\infty,-\sqrt{3})\cup(0,\sqrt{3})$ *and concave up on* $(-\sqrt{3},0)\cup(\sqrt{3},\infty)$.

(d) (3 pts.) List any inflection points, or if there aren't any state that there are no inflection points.

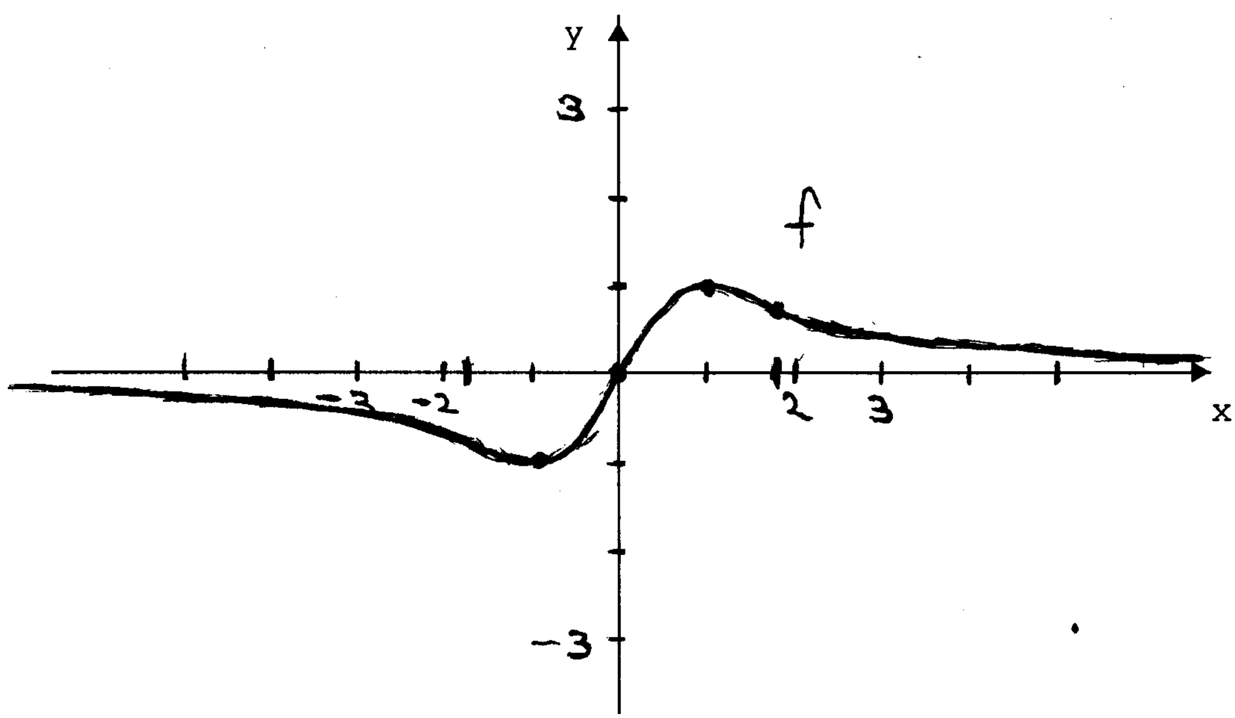
The inflection points of f *are* $(-\sqrt{3},-\frac{\sqrt{3}}{2})$, $(0,0)$, *and* $(\sqrt{3},\frac{\sqrt{3}}{2})$.

(e) (3 pts.) Locate any asymptotes.

$$\lim_{x\to\pm\infty} \frac{2x}{x^2+1} = 0.$$

Thus, y = 0 is a horizontal asymptote. There are no vertical asymptotes since f is defined everywhere on the real line.

(f) (5 pts.) Carefully sketch the graph of f below by plotting a few essential points and then connecting the dots appropriately.



Silly 10 point Bonus Problem: Show how to use the Mean Value Theorem to prove the following result: If f is continuous on a closed interval [a,b] with f′(x) > 0 for each x in (a,b), then f is increasing on the closed interval [a,b]. [Say where your work is, for it won't fit here!] **P. 4 of 5.**