Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

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1. (10 pts.) Here are five trivial limits to evaluate:

(a) $\lim_{x \to \infty} \frac{1}{\ln x} = 0$

(b) $\lim_{y \to -\infty} \frac{5y}{|y|} = \lim_{y \to -\infty} \frac{5y}{-y} = -5$

(c) $\lim_{h \to -\infty} \left(5 - \frac{1}{h}\right) = 5 - 0 = 5$

(d) $\lim_{x \to \infty} \frac{1}{2 + \sin(x)} =$ *This limit does not exist. Sinus Problem??*

(e) $\lim_{h \to \infty} (-2h) = -\infty$

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2. (15 pts.) Suppose that

$$h(x) = \begin{cases} 2x - x^2 & , \textit{if } x > 2 \\ 1 & , \textit{if } x = 2 \\ \frac{x^3 - 8}{3x^3} & , \textit{if } x < 2 \end{cases}$$

Evaluate each of the following easy limits.

(a) $\lim_{x \to \infty} h(x) = \lim_{x \to \infty} (2x - x^2) = \lim_{x \to \infty} (-x^2)\left(1 - \frac{2}{x}\right) = -\infty$

(b) $\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \left(\frac{x^3 - 8}{3x^3}\right) = \lim_{x \to -\infty} \left(\frac{1}{3} - \frac{8}{3x^3}\right) = \frac{1}{3}$

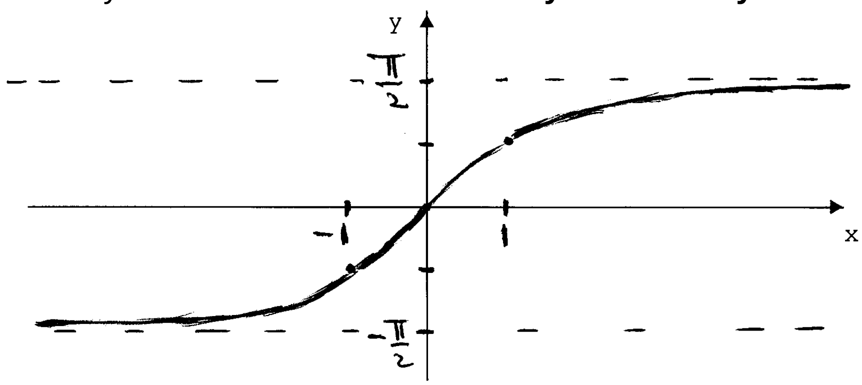
(c) $\lim_{x \to 1} h(x) = \lim_{x \to 1} \frac{x^3 - 8}{3x^3} = -\frac{7}{3}$

(d) $\lim_{x \to 2} h(x) = 0$ since

$\lim_{x \to 2^+} h(x) = \lim_{x \to 2^+} (2x - x^2) = 0$ and $\lim_{x \to 2^-} h(x) = \lim_{x \to 2^-} \frac{x^3 - 8}{3x^3} = 0.$

(e) $\lim_{x \to 3} h(x) = \lim_{x \to 3} (2x - x^2) = 2(3) - (3)^2 = -3$

3. (7 pts.) Carefully sketch the function $f(x) = \tan^{-1}(x)$ on the coordinate system below. **Label very carefully**.



Then evaluate the following two limits:

$$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2} \qquad \lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$

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4. (8 pts.) If $f(x) = x^2 + 6x$ and $h \neq 0$, then simplifying as much as possible allows us to write

$$\frac{f(x+h) - f(x)}{h} = \frac{[\,(x+h)^2 + 6(x+h)\,] - [\,x^2 + 6x\,]}{h} = \ldots = 2x + 6 + h$$

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5. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for

(*) $$\lim_{x \to a} f(x) = L.$$

Suppose that f is a function that is defined everywhere in some open interval containing x = a, except possibly at x = a. We write (*) if L is a number such that for each $\varepsilon > 0$ we can find a $\delta > 0$, such that if x is in the domain of f and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.//

(b) Using only the mathematical definition of limit, provide a complete proof that

(**) $$\lim_{x \to -3} \frac{x^2 - 9}{x + 3} = -6.$$

Proof: Let $\varepsilon > 0$ be arbitrary. Set $\delta = \varepsilon$. Observe that $\delta > 0$. Suppose now that x satisfies $0 < |x - (-3)| < \delta$. We now verify that

$0 < |x - (-3)| < \delta$ implies $|[(x^2 - 9)/(x + 3)] - (-6)| < \varepsilon$.

Now

$$\begin{aligned} 0 < |x - (-3)| < \delta \;&\Rightarrow\; |x + 3| < \varepsilon \\ &\Rightarrow\; |(x - 3) + 6| < \varepsilon \\ &\Rightarrow\; |(x - 3)(x + 3)/(x + 3) - (-6)| < \varepsilon \\ &\Rightarrow\; |[(x^2 - 9)/(x + 3)] - (-6)| < \varepsilon. \end{aligned}$$

Since,
given an arbitrary $\varepsilon > 0$, we have produced a number $\delta > 0$ such that, if x satisfies $0 < |x - 0| < \delta$, then $|[(x^2 - 9)/(x + 3)] - (-6)| < \varepsilon$, we have proved (**) above is true by applying the definition. [Without *scratching*!!]

6. (25 pts.) For each of the following, find the limit if the limit exists. If the limit fails to exist, say so. Be as precise as possible here. [Work on the back of Page 2 of 4 if you run out of room here.]

(a) $\lim_{x \to 1^-} \frac{x^2 - x - 2}{x^2 - 1} = \lim_{x \to 1^-} \frac{x-2}{x-1} = \infty$

since both $x - 1 < 0$ and $x - 2 < 0$ for $x < 1$.

(b) $\lim_{x \to -1^+} \frac{x^2 - x - 2}{x^2 - 1} = \lim_{x \to -1^+} \frac{x-2}{x-1} = \frac{3}{2}$

(c) $\lim_{x \to 9} \frac{9 - x}{x^{1/2} - 3} = \lim_{x \to 9} \frac{(9-x)(x^{1/2}+3)}{x-9} = \lim_{x \to 9}(-1)(x^{1/2}+3) = -6$

(d) $\lim_{x \to 2^-} \frac{4x - 8}{|x - 2|} = \lim_{x \to 2^-} \frac{4(x-2)}{2-x} = -4$

(e) $\lim_{x \to -1} (4 - 36x^2)^{1/5} = (-32)^{1/5} = -2$

Silly 10 point Bonus Problem: Provide an ε - δ proof that

$$\lim_{x \to 0} 4x \sin\left(\frac{1}{x}\right) = 0.$$

Hint: The *wiggle* can be controlled.

Proof: Let $\varepsilon > 0$ be arbitrary. Then set $\delta = \varepsilon/4$. Plainly $\delta > 0$. We shall now show that if x is any real number for which $0 < |x - 0| < \delta$, then

$$\left| 4x \sin\left(\frac{1}{x}\right) - 0 \right| < \varepsilon.$$

To this end, let x be any real number with $0 < |x - 0| < \delta$. Then

$$\begin{aligned} 0 < |x-0| < \delta \quad &\Rightarrow \quad |x| < \frac{\varepsilon}{4} \\ &\Rightarrow \quad 4|x| < \varepsilon \\ &\Rightarrow \quad \left|\sin\left(\frac{1}{x}\right)\right| |4x| < \varepsilon \\ &\Rightarrow \quad \left|4x \sin\left(\frac{1}{x}\right) - 0\right| < \varepsilon \end{aligned}$$

since

$$|\sin(1/x)| \le 1 \;\Rightarrow\; 4|x| \cdot |\sin(1/x)| \le 4|x| \quad \textit{when} \quad x \ne 0,$$

and for all real numbers a and b, we have $|ab| = |a| \cdot |b|$. //

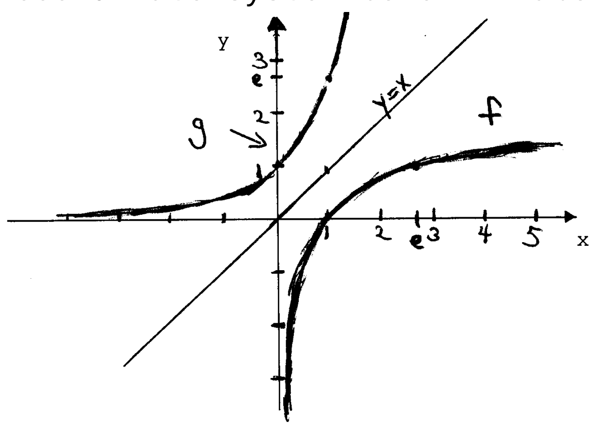
7. (10 pts.) Evaluate each of the following thorny limits:

(a) $$\lim_{x\to+\infty}\frac{\ln(2x)}{\ln(3x)} = \lim_{x\to+\infty}\frac{\ln(2)+\ln(x)}{\ln(3)+\ln(x)} = \lim_{x\to+\infty}\frac{\left(\frac{\ln(2)}{\ln(x)}+1\right)}{\left(\frac{\ln(3)}{\ln(x)}+1\right)} = 1$$

(b) $$\lim_{x\to 0}\frac{(x+4)^{1/2}-2}{x} = \lim_{x\to 0}\frac{x}{x[(x+4)^{1/2}+2]} = \ldots = \frac{1}{4}$$

The key piece of algebraic magic here is "rationalizing the numerator," done by multiplying by "1" in the appropriate form, the conjugate factor over itself.

8. (15 pts.) (a) (5 pts.) Carefully sketch both f(x) = ln(x) and g(x) = e^x on the coordinate system below. **Label very carefully**.



(b) (10 pts.) Evaluate each of the following limits.

$$\lim_{x\to 0^+}\ln(x) = -\infty \qquad \lim_{x\to\infty}\ln(x) = \infty$$

$$\lim_{x\to -\infty} e^x = 0 \qquad \lim_{x\to\infty} e^x = \infty$$

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x} = \lim_{u\to\infty}\left[\left(1+\frac{1}{u}\right)^u\right]^{\frac{3}{2}} = e^{3/2}$$ by substituting $u = 2x$.

Silly 10 point Bonus Problem: Provide an ε - δ proof that

$$\lim_{x\to 0} 4x\sin\left(\frac{1}{x}\right) = 0.$$

Say where your work is, for it won't fit here. Hint: The *wiggle* can be controlled. *An answer may be found on Page 3 of 4 at the bottom.*