Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.*

1. (25 pts.) Compute the derivatives of the following functions. You may use any of the rules of differentiation that are at your disposal. Do not attempt to simplify the algebra in your answers.

(a) $f(x) = 6x^4 - 12x^{-7} + 8\sec(x)$

$$f'(x) = 24x^3 + 84x^{-8} + 8\sec(x)\tan(x)$$

(b) $g(x) = (2x^2 - 4x^{-1})\tan(x)$

$$g'(x) = \left(4x + 4x^{-2}\right)\tan(x) + \left(2x^2 - 4x^{-1}\right)\sec^2(x)$$

(c) $h(t) = \dfrac{\sin(t)}{5t^{10}}$

$$h'(t) = \frac{5t^{10}\cos(t) - 50t^9\sin(t)}{25t^{20}}$$

(d) $y = csc(\cot(2\theta+1))$

$$\frac{dy}{d\theta} = -csc(\cot(2\theta+1))\cot(\cot(2\theta+1))(-1)(csc^2(2\theta+1))(2)$$

(e) $L(z) = \cos(4z^8) + 4\tan(\dfrac{\pi}{4}) - 4\sin(\dfrac{z}{2})$

$$\frac{dL}{dz}(z) = -\sin(4z^8)\cdot(32z^7) + 0 - 4\cos(\frac{z}{2})\cdot(\frac{1}{2})$$

Prove that if a and b are positive, then the equation

$$(*) \qquad \frac{a}{x-1} + \frac{b}{x-3} = 0$$

has at least one solution in the interval (1,3).

Solution B:

Let $f(x) = a/(x-1) + b/(x-3)$ for $1 < x < 3$.

Obviously,

$$f(x) = \frac{a(x-3) + b(x-1)}{(x-1)(x-3)}$$

by doing routine algebra. The numerator is zero precisely when

$$x = \frac{3a+b}{a+b}.$$

To show that this number is in the interval (1,3), we observe that if $a > 0$ and $b > 0$,

$$1 = \frac{a+b}{a+b} < \frac{3a+b}{a+b} < \frac{3a+3b}{a+b} = 3.$$

[This last is required to completely answer the question as it was asked.]

2. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition of **continuity** of a function f(x) at a point x = a.

A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

(b) Is there a real number k, that will make the function $f(x)$ defined below continuous at $x = 2$? Either find the value for k and using the definition of continuity, prove that it makes f continuous at $x = 2$, or explain why there cannot be such a number k. Suppose

$$f(x) = \begin{cases} kx^2 & , x \le 2 \\ 2x + k & , x > 2 \end{cases}$$

From the definition of continuity at a point, in order for f to be continuous at $x = 2$, it is necessary and sufficient for f to be both left and right continuous. Since

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x + k) = 4 + k$$

and

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} kx^2 = 4k = f(2) ,$$

f is continuous at $x = 2$ if, and only if $4k = 4 + k$. The only solution to this equation is $k = 4/3$.

3. (10 pts.) (a) Using complete sentences and appropriate notation, state the Intermediate Value Theorem.
// If f is continuous on a closed interval [a,b], and k is any number between f(a) and f(b), inclusive, then there is a number x_0 in the interval [a,b] with $f(x_0) = k$.//
(b) Prove that if a and b are positive, then the equation

$$(*) \qquad \frac{a}{x - 1} + \frac{b}{x - 3} = 0$$

has at least one solution in the interval (1,3).

First note that I didn't tie your hands behind your back, for I didn't require you use the Intermediate Value Theorem. That means there actually turn out to be two easy legitimate routes to a proof.

Solution A: Let $f(x) = a/(x\text{-}1) + b/(x\text{-}3)$ for $1 < x < 3$. Then f is continuous on the interval (1,3). Since

$$\lim_{x \to 1^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to 3^-} f(x) = -\infty ,$$

it follows that there are numbers x_1 and x_2 with $1 < x_1 < x_2 < 3$ and $f(x_1) > 0$ and $f(x_2) < 0$. Since f is continuous on $[x_1, x_2]$ and 0 is between $f(x_1)$ and $f(x_2)$, the Intermediate Value Theorem may be applied to see there is a number c and $f(c) = 0$, Of course c is a solution to (*).//

Solution B: This is at the bottom of Page 1 of 4.

4. (5 pts.) Evaluate the following limit:

$$\lim_{x \to +\infty} \sin^{-1}\left(\frac{x}{1 - 2x}\right) = \sin^{-1}\left(\lim_{x \to +\infty} \frac{x}{1 - 2x}\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

5. (5 pts.) Compute $f''(x)$ when

$$f(x) = \sin(3x^2).$$

$$f'(x) = \cos(3x^2)\cdot(6x) = 6x\cos(3x^2).$$

$$f''(x) = 6\cos(3x^2) - 6x\sin(3x^2)\cdot(6x) = 6\cos(3x^2) - 36x^2\sin(3x^2).$$

6. (5 pts.) Find an equation for the tangent line to the graph of

$$y = \tan(4x^2)$$

when $x = \frac{\sqrt{\pi}}{4}$.

Since $y(x) = \tan(4x^2)$ implies that $y'(x) = 8x\sec^2(4x^2)$, we have

$$y\left(\frac{\sqrt{\pi}}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1 \quad \text{and} \quad y'\left(\frac{\sqrt{\pi}}{4}\right) = 2\sqrt{\pi}\sec^2\left(\frac{\pi}{4}\right) = 4\sqrt{\pi} \ .$$

Tangent line equation: $y - 1 = 4\sqrt{\pi}\left(x - \frac{\sqrt{\pi}}{4}\right)$, *or* $y = 4\sqrt{\pi}\,x - \pi + 1$.

7. (10 pts.) (a) Find all values in the interval $[-\pi,\pi]$ at which the graph of f has a horizontal tangent line when $f(x) = x + \cos(x)$.

Since f has a horizontal tangent line when $f'(x) = 0$, and $f'(x) = 1 - \sin(x)$, it follows that f has a horizontal tangent line in the interval $[-\pi,\pi]$ when $\sin(x) = 1$. This happens only when $x = \pi/2$.

(b) Using that the following limit represents $f'(a)$ for some function f and some number a, evaluate it:

Here, of course, $f(x) = x^7$ and $a = 1$. So we have

$$\lim_{x\to 1}\frac{x^7 - 1}{x - 1} = f'(1) = 7(1)^6 = 7 \quad \textit{since } f'(x) = 7x^6.$$

8. (5 pts.) Given that the tangent line to $y = f(x)$ at the point (1,2) passes through the point (-1,-1), find $f'(1)$.

Since the numerical value of $f'(1)$ is merely the slope of the tangent line passing through the point of tangency, (1,2), we only need to compute it. Thus,

$$f'(1) = \frac{(-1) - (2)}{(-1) - (1)} = \frac{3}{2}.$$

9. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative, $f'(x)$, of a function $f(x)$.

// The function f′ defined by the equation

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

is called the derivative of f with respect to x. The domain of f′ consists of all x in the domain of f for which the limit above exists.//

(b) Using only the definition of the derivative as a limit, show all steps of the computation of $f'(x)$ when $f(x) = x^3$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to o} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\ &= \lim_{h\to 0} \left(3x^2+3xh + h^2\right) = 3x^2 \ \textit{for every real number } x. \end{aligned}$$

10. (5 pts.) Pretend f is a magical function that has the property that at x = 3 the tangent line f is actually defined by the equation y = -5(x - 1) + 2. Obtain

(a) $f(3) = -5(3 - 1) + 2 = -8$ (b) $f'(3) = -5$

11. (10 pts.) Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/hour. How fast is the radius of the spill increasing when the area is 9 square miles??

Let $A(t)$ denote the area, in square miles, of the spill as a function of time, in hours. Then $A(t) = \pi(r(t))^2$, where $r(t)$ is the radius at time t. At the time in question, say $t = t_0$,

$$6 = A'(t_0) = 2\pi(r(t_0))r'(t_0) = 2\pi\left(\frac{3}{\sqrt{\pi}}\right)\cdot r'(t_0)$$

since

$$9 = \pi(r(t_0))^2 \Rightarrow r(t_0) = \frac{3}{\sqrt{\pi}}.$$

Solving for $r'(t_0)$ yields

$$r'(t_0) = \frac{1}{\sqrt{\pi}} \ \textit{miles per hour}.$$

Silly 10 point Bonus Problem: Suppose that a function f is differentiable at x_0 and that $f'(x_0) > 0$. Prove that there exists an open interval containing x_0 such that if x_1 and x_2 are any two points in this interval with $x_1 < x_0 < x_2$, then $f(x_1) < f(x_0) < f(x_2)$. // Say where your work is for it won't fit here. Hint: Epsilon antics work wonders.

This is Problem #47 of Section 3.2. A solution is online at the Math Dept. Calculus I Help site in the Complete Solutions Manual for the text.