Silly 10 point Bonus Problem: Suppose that a function f is differentiable at x_0 and that $f'(x_0) > 0$. Prove that there exists an open interval containing x_0 such that if x_1 and x_2 are any two points in this interval with $x_1 < x_0 < x_2$, then $f(x_1) < f(x_0) < f(x_2)$. // Say where your work is for it won't fit here. Hint: Epsilon antics work wonders.

This is Problem #47 of Section 3.2. A solution is online at the Math Dept. Calculus I Help site in the Complete Solutions Manual for the text.

Here is a more leisurely discussion of this problem.

First recall that the existence of the derivative of f at x_0 is equivalent to the existence of the following limit:

$$(*)\qquad f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Consequently, if we assume that $f'(x_0) > 0$, we should expect that for each x that is near enough to x_0 that the quotient

$$Q(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

must be positive. What this entails now is this:

If x_1 is sufficiently near x_0 so that $Q(x_1) > 0$, and $x_1 < x_0$, then since $x_1 - x_0$ is negative, we must also have $f(x_1) - f(x_0)$ negative or $f(x_1) < f(x_0)$.

Similarly, if x_2 is sufficiently near x_0 so that $Q(x_2) > 0$, and $x_0 < x_2$, then since $x_2 - x_0$ is positive, we must also have $f(x_2) - f(x_0)$ positive or $f(x_0) < f(x_2)$.

The trick, such as it is now, is to use the definition of the limit defining the derivative of f at x_0 in terms of ε's and δ's to produce an open interval centered at x_0 where all this magic happens.

How do we proceed?? Suppose that $f'(x_0) > 0$. Let $\varepsilon > 0$. From the definition of the limit (*) above, it follows that there is a number $\delta > 0$ such that, for each x with $0 < |x - x_0| < \delta$, we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon,$$

or equivalently

$$(**)\qquad f'(x_0) - \varepsilon < \frac{f(x) - f(x_0)}{x - x_0} < f'(x_0) + \varepsilon, .$$

Look at the leftmost inequality in (**) above. Plainly $f'(x_0) - \varepsilon > 0$ precisely when $\varepsilon < f'(x_0)$. It turns out that a convenient, cute ε that does the job is $f'(x_0)/2$. The corresponding δ provides us the open interval $I = (x_0 - \delta, x_0 + \delta)$ where (**) is true with the leftmost member being positive. The magic sketched up the page is now true.

You may now want to look at the "Complete Solutions Manual" answer where the epsilon was chosen cutely. Read. e.t.