Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.

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1. (10 pts.) (a) (6 pts.) Find formulas for Δy and the differential dy when $y = x^3$. Label your expressions correctly.

Evidentally $dy = 3x^2dx$ is cheap thrills, but Δy is slightly messier.

$$\begin{aligned}\Delta y &= (x + \Delta x)^3 - x^3 \\ &= [\,x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3] - x^3 \\ &= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.\end{aligned}$$

(b) (4 pts.) Use an appropriate local linear approximation formula to estimate $(80.9)^{1/2}$.

Let $f(x) = x^{1/2}$ and set $x_0 = 81$. Then the local linear approximation at x_0 is given by

$$x^{1/2} \approx x_0^{1/2} + \frac{1}{2}x_0^{-1/2}(x - x_0) = 81^{1/2} + \frac{1}{2}\cdot\frac{1}{81^{1/2}}(x - 81)$$

for all x. When $x = 80.9$,

$$(80.9)^{1/2} \approx 81^{1/2} + \frac{1}{2}\cdot\frac{1}{81^{1/2}}(80.9 - 81) = 9 - \frac{1}{180} \approx 8.994.$$

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2. (10 pts.) (a) Use logarithmic differentiation to find dy/dx when

$$y = (\ln(x))^{\tan(x)}.$$

$$\begin{aligned} y = (\ln(x))^{\tan(x)} \quad &\Rightarrow \quad \ln(y) = \tan(x)\cdot\ln(\ln(x)) \\ &\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \sec^2(x)\ln(\ln(x)) + \tan(x)\frac{1}{x\,\ln(x)}.\end{aligned}$$

Thus,

$$\frac{dy}{dx} = \left[\sec^2(x)\ln(\ln(x)) + \tan(x)\frac{1}{x\,\ln(x)}\right](\ln(x))^{\tan(x)}.$$

(b) Find dy/dx by using implicit differentiation when

$$\sin(x^2y^2) = x.$$

First, pretend that y is a function of x. Then

$$\begin{aligned} \sin(x^2y^2) = x \quad &\Rightarrow \quad \frac{d}{dx}(\sin(x^2y^2)) = \frac{d}{dx}(x) \\ &\Rightarrow \quad \cos(x^2y^2)\left[2xy^2 + 2x^2y\frac{dy}{dx}\right] = 1 \\ &\Rightarrow \quad \frac{dy}{dx} = \frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)} \quad \textit{or equivalent}.\end{aligned}$$

Note: There is additional simplification that may be done above.

3. (10 pts.) Let

$$f(x) = x^5 + x^3 + x.$$

(a) Show that f is one-to-one on $\mathbb{R}$ and confirm the $f(1) = 3$.

Since $f'(x) = 5x^4 + 3x^2 + 1 \geq 1 > 0$ *for each* $x \in \mathbb{R}$, f is increasing, and thus one-to-one on $\mathbb{R}$. $f(1) = 1^5 + 1^3 + 1 = 3$.

(b) Find $(f^{-1})'(3)$.

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{5(1)^4 + 3(1)^2 + 1} = \frac{1}{9}.$$

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4. (10 pts.) Write down each of the following derivatives. [2 pts/part.]

(a) $\frac{d[\sec^{-1}]}{dx}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

(b) $\frac{d[\cos^{-1}]}{dx}(x) = \frac{-1}{\sqrt{1-x^2}}$

(c) $\frac{d[\cot^{-1}]}{dx}(x) = \frac{-1}{1+x^2}$

(d) $\frac{d[csc^{-1}]}{dx}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$

(e) $\frac{d[\sin^{-1}]}{dx}(x) = \frac{1}{\sqrt{1-x^2}}$

Silly 10 point Bonus Problem: Show

$$(*) \qquad e^x \geq 1+x \ \ \textit{if} \ \ x \geq 0.$$

Observe that if we set $h(x) = e^x - 1 - x$, then (*) is equivalent to

$$(**) \qquad h(x) \geq 0 \ \ \textit{if} \ \ x \geq 0.$$

Plainly $h(0) = 0$, and $h'(x) = e^x - 1$. Since $e^x > 1$ when $x > 0$, it follows that $h'(x) > 0$ when $x > 0$. Since h is continuous on the interval $[0,\infty)$, it follows that h is increasing on $[0,\infty)$. Thus, $0 < x$ implies $0 < h(x)$, and we are finished.

5. (10 pts.) Fill in the blanks appropriately.

(a) A function f is increasing on an interval (a, b) if $\underline{\quad f(x_1) < f(x_2) \quad}$

whenever $a < x_1 < x_2 < b$.

(b) A function f is decreasing on an interval (a, b) if $\underline{\quad f(x_1) > f(x_2) \quad}$

whenever $a < x_1 < x_2 < b$.

(c) A function f is concave down on an interval (a, b) if the derivative of f is $\underline{\text{ decreasing }\qquad\qquad}$ on (a, b).

(d) A function f is concave up on an interval (a, b) if the derivative of f is $\underline{\text{ increasing }\qquad\qquad}$ on (a, b).

(e) A function f has a relative minimum at x_0 if there is an open interval containing x_0 on which $\underline{\quad f(x_0) \le f(x) \quad}$ for each x in both the interval and the domain of f.

6. (5 pts.) Find the following limit by interpreting the expression as an appropriate derivative.

$$\lim_{h \to 0} \frac{\tan^{-1}(1+h) - \frac{\pi}{4}}{h} = \frac{d[\tan^{-1}]}{dx}(1) = \frac{1}{1+(1)^2} = \frac{1}{2}.$$

7. (5 pts.) The side of the cube is measured to be 10 ft, with a possible error of ±0.1 ft. Using differentials estimate the percent error in the calculated volume.

The volume of a cube is given by $V(x) = x^3$, where x is the length of a side. Using differentials, we may approximate the relative error as follows:

$$\frac{\Delta V}{V} \approx \frac{dA}{A} = \frac{3(x_0)^2 dx}{x_0^3} = \frac{3\Delta x}{x_0} = \frac{(3)(\pm 0.1)}{10} = \pm 0.03 = \pm 3\%$$

with $x_0 = 10$ and setting $dx = \Delta x = \pm 0.1$. Remember that the percent error is merely the relative error written as a percentage.

8. (8 pts.) Assume f is continuous everywhere. If

$$f'(x) = 4x^3 - 36x^2 = 4x^2(x - 9),$$

find all the critical points of f and at each stationary point apply the second derivative test to determine relative extrema, if possible. If the second derivative test fails at a critical point, apply the first derivative test to determine the true state of affairs there.

We may read off of the derivative that the critical points are $x = 0$ and $x = 9$. Since

$$f''(x) = 8x(x - 9) + 4x^2,$$

$f''(0) = 0$ and $f''(9) = 4(9)^2 > 0$. Thus, the second derivative test provides no information at $x = 0$, and implies that f has a relative minimum at $x = 9$.

When $x < 9$, $f'(x) < 0$ when $x \neq 0$. The first derivative test now implies that f has neither a local max at $x = 0$ nor a local min there.//

9. (12 pts.) Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible.

(a)

$$\lim_{x \to +\infty} 2x \sin\left(\frac{\pi}{x}\right) = \lim_{x \to +\infty} \frac{2\sin(\pi x^{-1})}{x^{-1}} \overset{(L'H)}{=} \lim_{x \to +\infty} \frac{2\cos(\pi x^{-1})(-\pi x^{-2})}{-x^{-2}}$$

$$= \lim_{x \to +\infty} 2\pi \cos\left(\frac{\pi}{x}\right) = 2\pi$$

(b) $$\lim_{x \to 0} \frac{\sin^{-1}(2x)}{x} \overset{(L'H)}{=} \lim_{x \to 0} \frac{2}{\sqrt{1-(2x)^2}} = 2$$

(c) $$\lim_{x \to 0} \frac{x - \tan^{-1}(x)}{x^3} \overset{(L'H)}{=} \lim_{x \to 0} \frac{1 - \dfrac{1}{x^2+1}}{3x^2} = \lim_{x \to 0} \frac{1}{3(x^2+1)} = \frac{1}{3}$$

Note that in (c) we have left some of the routine algebra to be filled in by the reader. Strictly speaking, the evaluation of limits (a) and (b) does not require the use of L'Hopital's Rule. These could have been handled readily using the results of Section 2.6 and a little algebraic prestidigitation.

10. (20 pts.) When f is defined by

$$f(x) = \frac{2}{x^2+1} \quad \textit{for } x \in \mathbb{R},$$

$$f'(x) = \frac{-4x}{(x^2+1)^2} \quad \text{and} \quad f''(x) = \frac{16(x-\frac{1}{2})(x+\frac{1}{2})}{(x^2+1)^3} .$$

(a) (3 pts.) What are the critical point(s) of f and what is the value of f at each critical point?

The only critical point of f *is* $x_1 = 0$. $f(0) = 2$.

(b) (3 pts.) Determine the open intervals where f is increasing or decreasing.

f *is decreasing on* $(0,\infty)$, *and increasing on* $(-\infty,0)$.

(c) (3 pts.) Determine the open intervals where f is concave up or concave down.

f *is concave down on* $(-1/2, 1/2)$ *and concave up on* $(-\infty, -1/2) \cup (1/2, \infty)$.

(d) (3 pts.) List any inflection points or state that there are none.

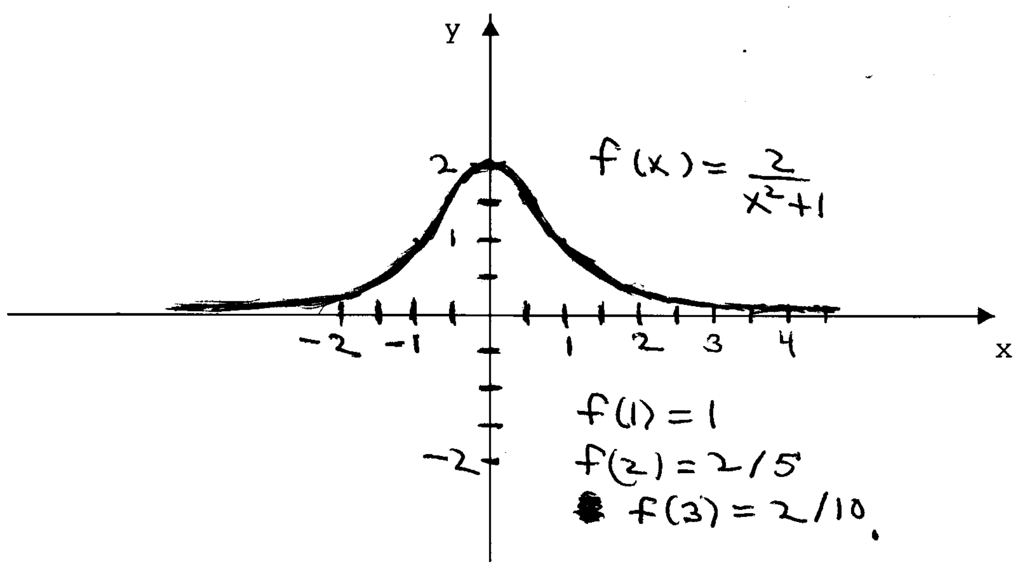
The inflection points of f *are* $(-1/2, 8/5)$ *and* $(1/2, 8/5)$.

(e) (3 pts.) Locate any asymptotes.

$$\lim_{x\to\pm\infty} \frac{2}{x^2+1} = 0.$$

Thus, y = 0 is a horizontal asymptote. There are no vertical asymptotes since f is defined everywhere on the real line.

(f) (5 pts.) Carefully sketch the graph of f below by plotting a few essential points and then connecting the dots appropriately.



Silly 10 point Bonus Problem: Show

$$e^x \geq 1+x \quad \textit{if} \quad x \geq 0.$$

[Say where your work is, for it won't fit here!] [On Page 2 of 5.]