Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

1. (20 pts.) Compute the derivatives of the following functions. You may use any of the rules of differentiation that are at your disposal. Do not attempt to simplify the algebra in your answers.

(a) $f(x) = 4x^{-5} - 3x^{6} + 7\cdot\cot(x)$

$f'(x) = -20x^{-6} - 18x^{5} - 7\csc^{2}(x)$

(b) $g(x) = (x^{3} - 4x^{-1})\sin(x)$

$g'(x) = (3x^{2} + 4x^{-2})\sin(x) - (x^{3} - 4x^{-1})\cos(x)$

(c) $h(t) = \dfrac{3\cot(t)}{2t^{5} - \sec(t)}$

$$h'(t) = \frac{-3\csc^{2}(t)(2t^{5} - \sec(t)) - (3\cot(t)(10t^{4} - \sec(t)\tan(t))}{(2t^{5} - \sec(t))^{2}}$$

(d) $L(x) = 4\cdot\tan^{2}(x^{3})$

$L'(x) = 8\cdot\tan(x^{3})\cdot\sec^{2}(x^{3})\cdot 3x^{2}$

(e) $y = \sin(21x) + 5\cos(7x) + \sec(\pi/4)$

$$\frac{dy}{dx} = 21\cos(21x) - 35\sin(7x)$$

Note: $\sec(\pi/4)$ is a constant.

2. (10 pts.) (a) Using complete sentences and appropriate notation, state the Intermediate Value Theorem.

If f is continuous on a closed interval [a,b], and k is any number between f(a) and f(b), inclusive, then there is a number x_0 in the interval [a,b] with $f(x_0) = k$.

(b) Can the Intermediate Value Theorem be used to show the equation

$$(2x - 1)/(x^2 - 4x - 5) = 0$$

has a solution in the interval [0,4]? Explain completely.

Let $f(x) = (x + 1)/(x^2 - 4x - 5) = (x + 1)/[(x-5)(x+1)]$ on [0,4]. Observe that f(0) = 1/5, f(4) = -7/5, and k = 0 is a number between f(0) and f(4). Clearly f is continuous on the interval [0,4], for the points where the rational function f fails to be defined lie outside of the interval. Thus, we can use the Intermediate Value Theorem to imply there is a number x_0 in [0,4] where $f(x_0) = 0$. Thus, the answer is yes.

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3. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative, f′(x), of a function f(x).

The function f′ defined by the equation

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

is called the derivative of f with respect to x. The domain of f′ consists of all x in the domain of f for which the limit above exists.

(b) Using the definition of the derivative as a limit, show all steps of the computation of f′(x) when $f(x) = 2x^2 - 4x$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[2(x+h)^2 - 4(x+h)] - [2x^2 - 4x]}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 - 4h}{h} \\ &= \lim_{h \to 0} (4x - 4 + 2h) \\ &= 4x - 4 \end{aligned}$$

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4. (10 pts.) Using complete sentences and appropriate notation, provide the precise mathematical definitions for each of the following items:

(a) $\lim_{x \to a} f(x) = L$ [Hint: This involves ε and δ.]//

Suppose that f is a function that is defined everywhere in some open interval containing x = a, except possibly at x = a. We write

$$\lim_{x \to a} f(x) = L$$

if L is a number such that for each $\varepsilon > 0$ we can find a $\delta > 0$, such that if x is in the domain of f and $0 < | x - a | < \delta$, then $| f(x) - L | < \varepsilon$.

(b) **Continuity** of a function f(x) at a point x = a //

A function f is continuous at x = a if

$$\lim_{x \to a} f(x) = f(a).$$

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5. (5 pts.) Find a value for the constant k, if possible, that will make the function h(x) defined below continuous at x = 5. If you find such a k, using the definition, verify the continuity of h(x) at x = 5. Suppose that

$$h(x) = \begin{cases} 2(x - 5)/(x^2 - 4x - 5), & x \neq 5 \\ 10 \cdot k, & x = 5. \end{cases}$$

In order for h to be continuous at x = 5, it is necessary and sufficient for

$$10 \cdot k = h(5) = \lim_{x \to 5} h(x) = \lim_{x \to 5} 2(x - 5)/(x^2 - 4x - 5) = \lim_{x \to 5} 2/(x+1)$$
$= 1/3$. Thus, $k = 1/30$.

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6. (5 pts.) Evaluate the following limit. To obtain full credit, you must show all essential steps correctly in a chain of equations.

$$\lim_{\theta \to 0} \frac{\tan(8\pi^3\theta)}{\sin(3\pi\theta)} = \lim_{\theta \to 0} \frac{\tan(8\pi^3\theta)}{8\pi^3\theta} \cdot \frac{3\pi\theta}{\sin(3\pi\theta)} \cdot \frac{8\pi^3}{3\pi\theta} = 8\pi^2/3$$

To get full credit, the intermediate steps must be provided, and they must be correct. The numerical answer by itself will not suffice here. [The first two factors above clearly have limit equal to one.]

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7. (5 pts.) Compute $f''(x)$ when $f(x) = \sin(4x^3)$. Label your expressions correctly or else.

$f'(x) = 12x^2 \cdot \cos(4x^3)$ [Observe that f' is a product!]

$f''(x) = 24x \cdot \cos(4x^3) - 12x^2 \cdot \sin(4x^3)(12x^2)$

$= 24x \cdot \cos(4x^3) - 144x^4 \cdot \sin(4x^3)$

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8. (5 pts.) Obtain an equation for the line tangent to the graph of $f(x) = \cos(2x)$ at $x_0 = \pi/6$.

$$y - (1/2) = -3^{1/2} \cdot (x - (\pi/6))$$

will do after the dust settles. [$f'(x) = -2\sin(2x)$ plus assorted trig treats are needed. For instance, $\sin(\pi/3) = 3^{1/2}/2$ and $\cos(\pi/3) = 1/2$ are useful pieces of trivia.]

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9. (5 pts.) Show an evaluation of the following limit that is completely correct. You will need to build a suitable inequality to provide a complete solution.

$$\lim_{x \to 0} x^2\sin(4/x^5) =$$

To see this, observe that for $x \neq 0$, $-1 \leq \sin(4/x^5) \leq 1$ implies that $-x^2 \leq x^2\sin(4/x^5) \leq x^2$. Since $x^2 \to 0$ as $x \to 0$, the squeezing theorem implies $x^2\sin(4/x^5) \to 0$ as $x \to 0$.

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10. (5 pts.) Find all points in the interval $[-2\pi,2\pi]$ where the graph of $f(x) = x - 2\cos(x)$ has a horizontal tangent line.

The graph of f has a horizontal tangent line where $f'(x) = 0$. Now $f'(x) = 1 + 2\sin(x)$. Thus $f'(x) = 0$ in the given interval where $\sin(x) = -1/2$. To deal with this, first find all solutions in the interval $[0,2\pi]$. This turns out to be at two points, $x_0 = 7\pi/6$, and $x_1 = 11\pi/6$. From 2π-periodicity, there are also two in the interval $[-2\pi,0]$: $x_2 = (7\pi/6) - 2\pi = -5\pi/6$, and $x_3 = (11\pi/6) - 2\pi = -\pi/6$.

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14. (5 pts.) Give an $\varepsilon - \delta$ proof that $\lim_{x \to 2}(7x - 2) = 12$.

Proof: Let $\varepsilon > 0$ be arbitrary. Set $\delta = \varepsilon/7$. Observe that $\delta > 0$. Suppose now that x satisfies $0 < |x - 2| < \delta$. We shall now verify that $0 < |x - 2| < \delta$ implies $|(7x - 2) - 12| < \varepsilon$. Now

$$\begin{aligned} 0 < |x - 2| < \delta \quad &\Rightarrow \quad |x - 2| < \varepsilon/7 \\ &\Rightarrow \quad 7|x - 2| < \varepsilon \\ &\Rightarrow \quad |7x - 14| < \varepsilon \\ &\Rightarrow \quad |(7x - 2) - 12| < \varepsilon. \end{aligned}$$

Since, given an arbitrary $\varepsilon > 0$, we have produced a number $\delta > 0$ such that, if x satisfies $0 < |x - 2| < \delta$, then $|(7x-2) - 12| < \varepsilon$, we have proved that $(8x - 2) \to 14$ as $x \to 2$. [Without *scratching*.]

11. (5 pts.) Obtain the following limit. This is easy if you grok the definition of the derivative.

$$\lim_{h \to 0} \frac{\sec((\pi/3) + h) - \sec(\pi/3)}{h} = (\sec)'(\pi/3)$$

$$= \sec(\pi/3) \cdot \tan(\pi/3)$$

$$= 2 \cdot 3^{1/2}$$

12. (5 pts.) Pretend f is a magical function that has the property that at $x = \pi$ the tangent line to the graph of f is actually defined by the equation $y = -\pi x + 3\pi^2$. Obtain the following: (a) $f'(\pi) = -\pi$ (b) $f(\pi) = 2\pi^2$

13. (5 pts.) Is the function g defined below differentiable at x = 0 ?? Prove your guess is correct.

$$g(x) = \begin{cases} (x - 1)^2 & , \quad x \geq 0 \\ -2x + 1 & , \quad x < 0 \end{cases}$$

There are two natural ways to tackle this problem. You could simply try to use the definition of the derivative as a limit. In doing that, you would have to deal with the critical limit as two one-sided limits because of the way the function g is defined. Alternatively, you might be tempted to use the theorem on page 199 that deals with just this sort of situation. Below, I'll show you how the funny theorem solves the problem.

First, $g(0) = (0 - 1)^2 = 1$,

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-2x + 1) = 1, \text{ and}$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x - 1)^2 = 1.$$

Thus, g is continuous at x = 0. This means that we have dealt with a significant chunk of the hypotheses of the magic theorem. Since

$$\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} -2 = -2, \text{ and}$$

$$\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} 2(x - 1) = -2,$$

the funny theorem now implies that $g'(0) = -2$. Oh joy!!

14. (5 pts.) Give an $\varepsilon - \delta$ proof that $\lim_{x \to 2}(7x - 2) = 12$.

[This proof appears on Page 4 below Problem 10.]