
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page.*

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1. (10 pts.) (a) Using implicit differentiation, compute dy/dx and d^2y/dx^2 when $x^3 + y^2 = 9$. **Label your expressions correctly or else.**

$d(x^3 + y^2)/dx = d(9)/dx$ implies that $dy/dx = -3x^2/(2y)$. Consequently, after differentiating one more time using quotient rule, replacing the occurance of dy/dx in the expression for the second derivative, and cleaning up the algebra, we obtain $d^2y/dx^2 = -[24xy^2 + 18x^4]/[8y^3] = -[12xy^2 + 9x^4]/[4y^3]$.

(b) Obtain an equation for the line tangent to the graph of $x^3 + y^2 = 9$ at the point $(-1,-(10)^{1/2})$.

Evaluating the implicit derivative above at $(-1,-(10)^{1/2})$ provides us with the slope of the tangent line, namely $3/(2(10)^{1/2})$. Consequently, an equation for the tangent line at the desired point is $y -(-(10)^{1/2}) = 3/(2(10)^{1/2})(x - (-1))$.

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2. (5 pts.) A 5-ft. ladder is leaning against the wall. If the top of the ladder slips down the wall at a rate of 4 ft./sec., how fast will the foot be moving away from the wall when the top is 3 ft. above the ground?

Let $x(t)$ denote the distance from the foot of the ladder to the base of the wall, and let $y(t)$ denote the vertical distance from the top of the ladder to the ground. From the Pythagorean Theorem, we have

(*) $$x^2(t) + y^2(t) = 5^2.$$

If t_0 denotes the time when the y is 3 ft., what we want is the value of $x'(t_0)$. Now $y(t_0) = 3$, (*) above, and 'x' being a distance imply that $x(t_0) = 4$. An implicit differentiation and a little obvious algabraic magic yield

$$x'(t_0) = -y(t_0)y'(t_0)/x(t_0).$$

Since $y'(t) = -4$ ft./sec., $x'(t_0)= (3)(4)/4 = 3$ feet per second. [Note: $y'(t) < 0$ since the distance from the top of the ladder to the ground is decreasing!]

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3. (5 pts.) Use logarithmic differentiation to find dy/dx when $y = x^{\sec(x)}$. **Label your expressions correctly or else.**

Since $\ln(y) = \sec(x)\ln(x)$, by doing an implicit differentiation and transposing y, we get

$$y' = [\sec(x)\cdot(1/x) + \sec(x)\tan(x)\ln(x)]x^{\sec(x)}.$$

4. (15 pts.) Differentiate the following functions. Do not attempt to simplify the algebra.

(a) $f(x) = \exp(3x^3 - 7x) - 2\cdot\ln(5x^3 - 14)$

$f'(x) = (9x^2 - 7)\exp(3x^3 - 7x) - 2(15x^2)/(5x^3 - 14)$

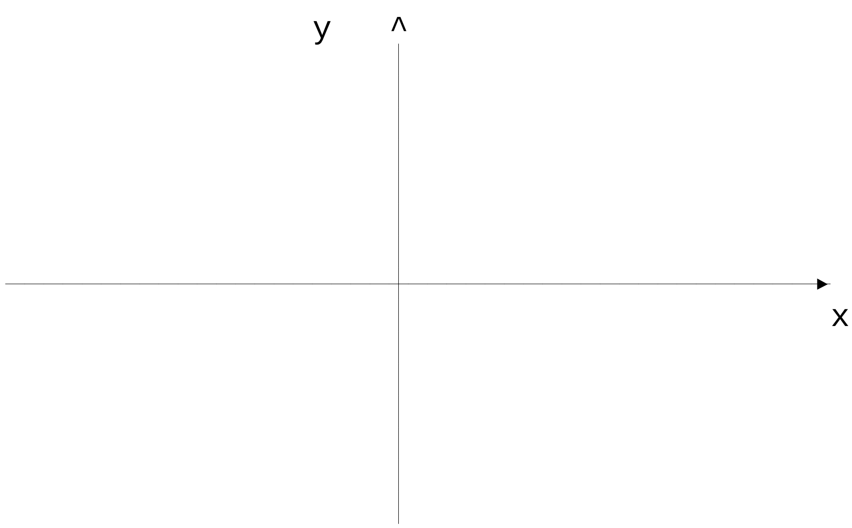
(b) $g(x) = 10^x + x^{10} + 10^{10} + \log_{10}(x) + \ln(10)$

$g'(x) = \ln(10)10^x + 10x^9 + 0 + 1/(x\cdot\ln(10)) + 0$

(c) $h(x) = \sec^{-1}(7x) + e^x\cdot\tan^{-1}(x) - 4\cdot\cos^{-1}(x^2)$

$h'(x) = 7/[|7x|((7x)^2-1)^{1/2}]+e^x\cdot\tan^{-1}(x)+e^x(1+x^2)^{-1} + 8x(1-x^4)^{-1/2}$

5. (5 pts.) Carefully sketch the graph of $y = \tan^{-1}(x)$. Label very carefully. [*This may be viewed at c1-t3-g.htm.*]



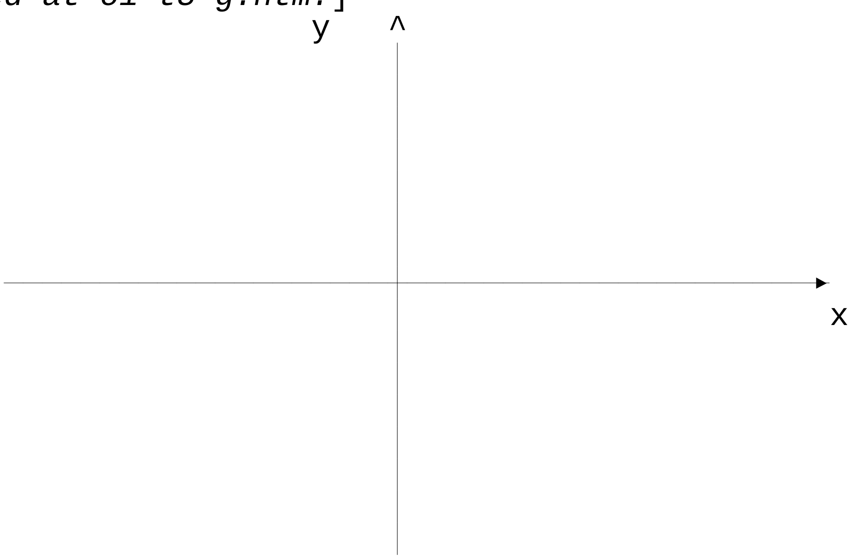
6. (5 pts.) Solve for x without using a calculator. Use the natural logarithm when logarithms are needed.

$$e^{2x} - e^{x} = 12 \iff e^{2x} - e^{x} - 12 = 0$$
$$\iff (e^{x} + 3)(e^{x} - 4) = 0$$
$$\iff e^{x} + 3 = 0 \text{ or } e^{x} - 4 = 0$$
$$\iff e^{x} = -3 \text{ or } e^{x} = 4$$
$$\iff x = \ln(4)$$

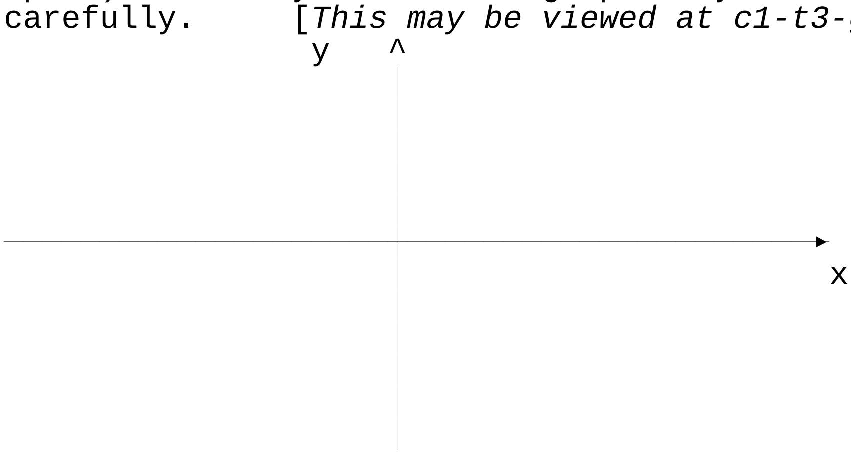
7. (5 pts.) Using a complete sentence and appropriate notation, provide the precise mathematical definitions for the following term: // The differential, dy, of a function f(x) //

If f is differentiable, dy, the differential, is defined by the equation dy = f′(x)dx, where dx is treated as an independent variable.

8. (5 pts.) Carefully sketch both $f(x) = \ln(x)$ and $g(x) = e^{x}$ on the coordinate system below. **Label very carefully**. [*This may be viewed at c1-t3-g.htm.*]



9. (5 pts.) Carefully sketch the graph of $y = \sin^{-1}(x)$. Label very carefully. [*This may be viewed at c1-t3-g.htm.*]



10. (10 pts.) Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible.

(a) $\lim_{t \to \infty} 7t \cdot \sin(3\pi \cdot t^{-1}) = 21\pi$

after writing

$$7t \cdot \sin(3\pi \cdot t^{-1}) = \frac{7 \cdot \sin(3\pi \cdot t^{-1})}{t^{-1}},$$

applying L'Hopital's Rule for 0/0 forms once, and cancelling the obvious common factor from the numerator and denominator.

[Note: A common mistake here is writing 7t as $1/(7t^{-1})$. The correct version is $1/(7t)^{-1}$, which will eventually lead to the correct result.]

(b) $\lim_{x \to 0} \dfrac{4 - 4 \cdot \cos(10x)}{e^{2x} + e^{-2x} - 2} = 400/8 = 50$

by applying L'Hopital's Rule for 0/0 forms twice.

[Note: A common mistake here is one of not using chain rule correctly in doing the differentiations needed.]

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11. (5 pts.) The radius of a sphere is measured to be 15 feet with a possible error of ±0.5 feet. Use differentials to estimate the relative error in the computed volume.

Set $V(r) = (4\pi/3)r^3$. The relative error in the computed volume is $\Delta V/V$. We shall approximate this using the differential dV. Thus, we have

$\Delta V/V \approx dV/V = 4\pi r^2 \cdot dr/[(4\pi/3)r^3] \approx (3/r) \cdot \Delta r = (1/5) \cdot (\pm 0.5)$

$= \pm 0.1$ when $r = 15$ and $\Delta r = \pm 0.5$.

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12. (5 pts.) Find the exact value of $\tan[2 \cdot \tan^{-1}(4/5)]$. [**Warning**: You will have to use several identities to handle this.]

$\tan[2 \cdot \tan^{-1}(4/5)] = 2 \cdot \tan[\tan^{-1}(4/5)]/[1 - \tan^2[\tan^{-1}(4/5)]$

$= 2 \cdot (4/5)/[1 - (4/5)^2]$

$= (8/5)/(9/25) = 40/9.$

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13. (5 pts.) Let $f(x) = \tan^{-1}(x^3) + (7\pi/4)\cdot x$. (a) Show f is invertible. (b) Then solve the equation $f^{-1}(x) = -1$.

(a) For each $x \in \mathbb{R}$ we have $f'(x) = [3x^2/(1+x^6)]+[7\pi/4] \geq 7\pi/4$. Thus $f'(x) > 0$ on $\mathbb{R} = (-\infty,\infty)$. It follows that f has an inverse.

(b) $f^{-1}(x) = -1 \quad \Leftrightarrow \quad x = f(-1)$
$\Leftrightarrow \quad x = \tan^{-1}(-1) - [7\pi/4]$
$\Leftrightarrow \quad x = -2\pi$

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14. (5 pts.) Use differentials and a linear approximation formula to estimate $(9)^{1/3}$. [Hint: Use $x_0 = 8$ and $f(x) = x^{1/3}$.]

Let $x_0 = 8$. Then $x_0 + \Delta x = 9$ implies that $\Delta x = 1$. Consequently,

$(9)^{1/2} \quad = f(x_0 + \Delta x) \quad \approx \quad f(x_0) + f'(x_0)\Delta x \quad = 2 + (1/12)$

$= 25/12$ since $f'(x_0) = (1/3)\cdot(8)^{-2/3} = 1/12$.

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15. (5 pts.) Solve for x without using a calculator.

$$\ln(64x) - 3\cdot\ln(x^2) = \ln(2)$$

Using standard properties of the natural log function, you can obtain $x = 32^{1/5} = 2$.

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16. (5 pts.) Carefully sketch the graph of $y = \cos^{-1}(x)$. Label very carefully. [*This may be viewed at c1-t3-g.htm.*]