Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.

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1. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition of **continuity** of a function f(x) at a point x = a.

A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

(b) Is there a real number k, that will make the function $f(x)$ defined below continuous at $x = 0$? Either find the value for k and using the definition of continuity, prove that it makes f continuous at $x = 0$, or explain completely why there cannot be such a number k. Suppose

$$f(x) = \begin{cases} \dfrac{\sin(x)}{|x|} & , \ x \neq 0 \\ k & , \ x = 0 \end{cases}$$

Since

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin(x)}{|x|} = \lim_{x \to 0^+} \frac{\sin(x)}{x} = 1$$

and

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin(x)}{|x|} = \lim_{x \to 0^-} \frac{\sin(x)}{-x} = -1 ,$$

$\lim_{x \to 0} f(x)$ doesn't exist. Consequently, there is no value of k that can make f continuous at $x = 0$.

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2. (10 pts.) (a) Using complete sentences and appropriate notation, state the Intermediate Value Theorem.

// If f is continuous on a closed interval [a,b], and k is any number between f(a) and f(b), inclusive, then there is a number x_0 in the interval [a,b] with $f(x_0) = k$.//

(b) Show that the equation $x^3 + x^2 - 2x = 1$ has at least one solution in the interval [-1,1].

Let $f(x) = x^3 + x^2 - 2x$ on [-1,1]. Observe that $f(-1) = 2$, $f(1) = 0$, and $k = 1$ is a number between $f(-1)$ and $f(1)$. Since f is a polynomial, f is continuous on the interval [-1,1]. Consequently, the function f satisfies the hypotheses of the Intermediate Value Theorem. Thus, we are entitled to invoke the magical conclusion that asserts that there is at least one number x_0 in (-1,1) where $f(x_0) = 1$.

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3. (5 pts.) Pretend f is a magical function that has the property that at x = 3 the tangent line f is actually defined by the equation y = -2(x - 1) + 5. Obtain

(a) $f(3) = -2(3 - 1) + 5 = 1$ (b) $f'(3) = -2$

4. (25 pts.) Compute the first derivatives of the following functions. You may use any of the rules of differentiation that are at your disposal. Do not attempt to simplify the algebra in your answers.

(a) $f(x) = 4x^6 - 7x^{-12} + 8\tan(x)$

$f'(x) = 24x^5 + 84x^{-13} + 8\sec^2(x)$

(b) $g(x) = (4x^2 - 2x^{-1})\sec(x)$

$g'(x) = (8x + 2x^{-2})\sec(x) + (4x^2 - 2x^{-1})\sec(x)\tan(x)$

(c) $h(t) = \dfrac{5t^{10} + 1}{\sin(t) + 2}$

$$h'(t) = \frac{(50t^9)(\sin(t) + 2) - (5t^{10} + 1)\cos(t)}{(\sin(t) + 2)^2}$$

(d) $y = \cot^5(2\theta + 1)$

$$\frac{dy}{d\theta} = 5(\cot(2\theta + 1))^4(-csc^2(2\theta + 1))(2) = -10\cot^4(2\theta + 1)csc^2(2\theta + 1)$$

(e) $L(z) = \sin(4z^8) + 4csc(\frac{\pi}{6}) - 4\cos(\frac{z}{2})$

$$\frac{dL}{dz}(z) = \cos(4z^8)(32z^7) + 0 - 4(-\sin(\frac{z}{2}))(\frac{1}{2}) = 32z^7\cos(4z^8) + 2\sin(\frac{z}{2})$$

Silly 10 point Bonus Problem: (Solution) Theorem 2.5.5 asserts that if g is a function with $\lim_{x \to c} g(x) = L$ and f is a function that is continuous at L, then

$$\lim_{x \to c} f(g(x)) = f(L).$$

It follows from $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1.$ that the function f defined by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & , \ x \neq 0 \\ 1 & , \ x = 0 \end{cases}$$

is continuous at $x = 0$. Since sine is continuous and one-to-one on $[-\pi/2, \pi/2]$, $\sin^{-1}$ is one-to-one and continuous on $[-1,1]$, and thus,

$$\lim_{x \to 0} \sin^{-1}(x) = \sin^{-1}(0) = 0.$$

Consequently, applying Theorem 2.5.5 with f as above, $g(x) = \sin^{-1}(x)$ and $c = 0$, since $\sin^{-1}(x) \neq 0$ when $x \neq 0$, we have

$$\lim_{x \to 0} \frac{x}{\sin^{-1}(x)} = \lim_{x \to 0} \frac{\sin(\sin^{-1}(x))}{\sin^{-1}(x)} = \lim_{x \to 0} f(g(x)) = f(0) = 1.$$

5. (10 pts.) What are the x- and y - intercepts of the tangent line to the graph of $y = 1/x^2$ at the point (2,1/4)?

The slope of the tangent line at x = 2 is

$$\left.\frac{dy}{dx}\right|_{x=2} = \left.\left(-\frac{2}{x^3}\right)\right|_{x=2} = -\frac{1}{4}.$$

Consequently, with a little routine algebra, we see that an equation for the tangent line in slope-intercept form is given by

$$y = -\frac{1}{4}x + \frac{3}{4}.$$

The y intercept is plainly $y = 3/4$ and one can see with a little work that the x intercept is $x = 3$.//

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6. (10 pts.) (a) Find all values in the interval $[0,2\pi]$ at which the graph of f has a horizontal tangent line when $f(x) = x + 2\cdot\cos(x)$.

Since f has a horizontal tangent line when $f'(x) = 0$, and $f'(x) = 1 - 2\sin(x)$, it follows that f has a horizontal tangent line in the interval $[0,2\pi]$ when $\sin(x) = 1/2$. This happens only at $x_1 = \pi/6$ or $x_2 = 5\pi/6$.

(b) The following limit represents $f'(a)$ for some function f and some number a. Using that information, evaluate the limit.

$$\lim_{x\to\pi}\frac{\sin(3x) - 0}{x - \pi} = -3.$$

Here, of course, $f(x) = \sin(3x)$ and $a = \pi$. So we have

$$\lim_{x\to\pi}\frac{\sin(3x) - 0}{x - \pi} = f'(\pi) = 3\cos(3\pi) = 3(-1) = -3 \quad \textit{since } f'(x) = 3\cos(3x).$$

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7. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative, $f'(x)$, of a function $f(x)$.// The function f' defined by the equation

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}.$$

is called the derivative of f with respect to x. The domain of f' consists of all x in the domain of f for which the limit above exists.//

(b) Using only the definition of the derivative as a limit, show all steps of the computation of $f'(x)$ when $f(x) = 1/x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to o}\frac{(x+h)^{-1} - x^{-1}}{h} \\ &= \lim_{h\to 0}\frac{x - (x+h)}{hx(x+h)} \\ &= \lim_{h\to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2} \quad \textit{for every real number } x \neq 0. \end{aligned}$$

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8. (5 pts.) Determine whether the following function is differentiable at x = 1.

$$f(x) = \begin{cases} x^2+x+2 & , x \le 1 \\ 3x & , x > 1 \end{cases}$$

Since

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3x = 3$$

and

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+x+2) = 4 ,$$

f is not continuous at $x = 1$. Consequently, f is cannot be differentiable there either. [Check continuity first, folks. There was a red herring here!]

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9. (5 pts.) Compute f″(x) when

$$f(x) = \sin(2x^3).$$

$$f'(x) = \cos(2x^3)\cdot(6x^2) = 6x^2\cos(2x^3).$$

$$f''(x) = 12x\cos(2x^3) - 6x^2\sin(2x^3)\cdot(6x^2) = 12x\cos(2x^3) - 36x^4\sin(2x^3).$$

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10. (10 pts.) A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm.? [$V = (4/3)\pi r^3$??]

Let $r(t)$ denote the radius of the sphere at time t in minutes. Then the volume is given by $V(t) = (4/3)\pi(r(t))^3$. We are told that $r'(t) = -15$ (cm/min) for any t. The question is, what is $V'(t_0)$ at the moment t_0 when $r(t_0) = 9$ cm? Evidently, $V'(t) = 4\pi(r(t))^2 r'(t)$ for any t. Substituting t_0 into this last equation, using the know values of r and r' at t_0, and doing the boring multiplication by hand yields $V'(t_0) = -4860\pi$ cubic cm/min.

Silly 10 point Bonus Problem: Explain completely how to obtain the limit

$$\lim_{x \to 0} \frac{x}{\sin^{-1}(x)} = 1$$

from

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Say where your work is, for it won't fit here. (Found on the bottom of Page 2 of 4.)