Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.

1. (10 pts.) (a) (6 pts.) Find formulas for Δy and the differential dy when $y = x^2 - 2x + 1$. Label your expressions correctly.

Evidentally $dy = (2x - 2)dx$ is cheap thrills, but Δy is slightly messier.

$$\begin{aligned}\Delta y &= [\,(x+\Delta x)^2 - 2(x+\Delta x) + 1\,] - [\,x^2 - 2x + 1\,]\\ &= [\,x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1\,] - [\,x^2 - 2x + 1\,]\\ &= (2x-2)\Delta x + (\Delta x)^2.\end{aligned}$$

(b) (4 pts.) Use an appropriate local linear approximation formula to estimate $(3.02)^4$.

Let $f(x) = x^4$ and set $x_0 = 3$. Then the local linear approximation at x_0 is given by

$$x^4 \approx x_0^4 + 4x_0^3(x - x_0) = 3^4 + 108(x - 3)$$

for all x. When $x = 3.02$,

$$(3.02)^4 \approx 81 + 108(0.02) = 81 + 2.16 = 83.16.$$

[True value: 83.181696.]

2. (10 pts.) (a) Use logarithmic differentiation to find dy/dx when

$$y = (\,x^3 - 2x\,)^{\ln(x)}$$

$$y = (x^3 - 2x)^{\ln(x)} \;\Rightarrow\; \ln(y) = \ln(x)\cdot\ln(x^3 - 2x)$$

$$\Rightarrow\; \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(x^3 - 2x) + \ln(x)\frac{3x^2 - 2}{x^3 - 2x}.$$

Thus,

$$\frac{dy}{dx} = \left[\frac{1}{x}\ln(x^3 - 2x) + \ln(x)\frac{3x^2 - 2}{x^3 - 2x}\right](x^3 - 2x)^{\ln(x)}.$$

(b) Find dy/dx by using implicit differentiation when

$$\cos(xy) = y.$$

First, pretend that y is a function of x. Then

$$\cos(xy) = y \;\Rightarrow\; \frac{d}{dx}(\cos(xy)) = \frac{dy}{dx}$$

$$\Rightarrow\; -\sin(xy)\left[y + x\frac{dy}{dx}\right] = \frac{dy}{dx}$$

$$\Rightarrow\; \frac{dy}{dx} = \frac{-y\sin(xy)}{1 + x\sin(xy)} \quad \textit{or equivalent}.$$

3. (10 pts.) (a) Let

$$f(x) = e^{x^3+x}$$

Show that f is one-to-one on $\mathbb{R}$ using f'. [A little explanation is needed!]

Since $f'(x) = (3x^2+1)e^{x^3+x} > 0$ *for each* $x \in \mathbb{R}$,
f is increasing, and thus one-to-one on $\mathbb{R}$.

(b) Suppose that a one-to-one function f has a tangent line given by $y = 5x + 3$ at the point (1,8). Find $f^{-1}(8)$ and $(f^{-1})'(8)$.

$$f^{-1}(8) = 1 \text{ and } (f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

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4. (10 pts.) Write down each of the following derivatives. [2 pts/part.]

(a) $\dfrac{d[\tan^{-1}]}{dx}(x) = \dfrac{1}{1+x^2}$

(b) $\dfrac{d[\sin^{-1}]}{dx}(x) = \dfrac{1}{\sqrt{1-x^2}}$

(c) $\dfrac{d[csc^{-1}]}{dx}(x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$

(d) $\dfrac{d[\cos^{-1}]}{dx}(x) = \dfrac{-1}{\sqrt{1-x^2}}$

(e) $\dfrac{d[\cot^{-1}]}{dx}(x) = \dfrac{-1}{1+x^2}$

Silly 10 point Bonus Problem: Show

(*) $\ln(x+1) \le x$ *if* $x \ge 0$.

Observe that if we set $h(x) = x - \ln(x + 1)$, then (*) is equivalent to

(**) $h(x) \ge 0$ *if* $x \ge 0$.

Plainly $h(0) = 0 - \ln(1) = 0$, and $h'(x) = 1 - (1/(x + 1)) = x/(x + 1)$. Thus, it follows that $h'(x) > 0$ when $x > 0$. Since h is continuous on the interval $[0,\infty)$, it follows that h is increasing on $[0,\infty)$. Thus, $0 < x$ implies $0 = h(0) < h(x)$, and we are finished.

5. (10 pts.) Fill in the blanks appropriately. [DEFINITIONS!!!]

(a) A function f has a relative maximum at x_0 if there is an open interval containing x_0 on which $\underline{\quad f(x_0) \geq f(x) \quad}$ for each x in both the interval and the domain of f.

(b) A function f is concave down on an interval (a, b) if the derivative of f is $\underline{\quad \text{decreasing} \quad}$ on (a, b).

(c) A function f is concave up on an interval (a, b) if the derivative of f is $\underline{\quad \text{increasing} \quad}$ on (a, b).

(d) A function f is decreasing on an interval (a, b) if $\underline{\quad f(x_1) > f(x_2) \quad}$ whenever $a < x_1 < x_2 < b$.

(e) A function f is increasing on an interval (a, b) if $\underline{\quad f(x_1) < f(x_2) \quad}$ whenever $a < x_1 < x_2 < b$.

6. (5 pts.) Find the following limit by interpreting the expression as an appropriate derivative.

$$\lim_{w \to 2} \frac{3\sec^{-1}(w) - \pi}{w - 2} = (3\sec^{-1})'(2) = \frac{3}{|2|\sqrt{2^2 - 1}} = \frac{\sqrt{3}}{2}$$

7. (5 pts.) Suppose the side of a square is measured to be 10 inches with a possible error of ± 1/32 inch. Estimate the error in the computed area of the square by using differentials. [You may leave your result as a fraction. You do not have to convert to decimal form.]

The area of a square is given by $A(x) = x^2$, where x is the length of a side in inches. Using differentials, we may approximate the computed error as follows:

$$\Delta A \approx dA = 2(x_0)dx = 2(10)\Delta x = (20)(\pm\frac{1}{32}) = \pm\frac{5}{8} \textit{ square inches}$$

with $x_0 = 10$ and setting $dx = \Delta x = \pm 1/32$.

It might be worth observing that

$$-\frac{5}{8} \leq \Delta A = 2(x_0)\Delta x + (\Delta x)^2 \leq \frac{5}{8} + (\frac{1}{32})^2,$$

$$\textit{since } 0 \leq (\Delta x)^2 \leq (\frac{1}{32})^2.$$

This might give you a sense of what is being ignored by the local linear approximation.

8. (8 pts.) Assume f is continuous everywhere. If

$$f'(x) = x(x - 2)^4$$

find all the critical points of f and at each stationary point apply the second derivative test to determine relative extrema, if possible. If the second derivative test fails at a critical point, apply the first derivative test to determine the true state of affairs there.

We may read off of the derivative that the critical points are $x = 0$ and $x = 2$. Since

$$f''(x) = (x - 2)^4 + 4x(x - 2)^3,$$

$f''(0) = 2^4$ and $f''(2) = 0$. Thus, the second derivative test provides no information at $x = 2$, and implies that f has a relative minimum at $x = 0$.

When $x > 0$, $f'(x) > 0$ when $x \neq 2$. The first derivative test now implies that f has neither a local max at $x = 2$ nor a local min there.//

9. (12 pts.) Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible.

(a) $$\lim_{x \to \pi^+} \frac{\sin(x)}{x - \pi} \overset{(L'H)}{=} \lim_{x \to \pi^+} \frac{\cos(x)}{1} = \cos(\pi) = -1$$

Observe that L'Hopital's really isn't needed here since this really ought to be a recognized derivative!

(b) $$\lim_{x \to +\infty} \left(1 - \frac{3}{x}\right)^x = \lim_{x \to +\infty} e^{x \ln\left(1 - \frac{3}{x}\right)} = e^{-3}$$

since the exponential function is continuous, and

$$\lim_{x \to +\infty} x \ln(1 - 3x^{-1}) = \lim_{x \to +\infty} \frac{\ln(1 - 3x^{-1})}{x^{-1}}$$

$$(L'H + algebra)$$

$$= \lim_{x \to +\infty} \frac{-3}{1 - \frac{3}{x}} = -3.$$

(c)

$$\lim_{x \to 0} \frac{x - \tan(x)}{x^3} \overset{(L'H)}{=} \lim_{x \to 0} \frac{1 - \sec^2(x)}{3x^2}$$

$$\overset{(L'H)}{=} \lim_{x \to 0} \frac{-2\sec(x)\tan(x)\sec(x)}{6x}$$

$$= \lim_{x \to 0} \left(-\frac{1}{3}\sec^2(x)\left(\frac{\tan(x)}{x}\right)\right)$$

$$= -\frac{1}{3}.$$

You may need an additional application of the magic theorem if you do not recognize the $\tan(x)/x$ piece of the puzzle.

10. (20 pts.) When f is defined by

$$f(x) = 2x^4 - 4x^2 = 2x^2(x+\sqrt{2})(x-\sqrt{2}) \ \ for\ x \in \mathbb{R},$$

$$f'(x) = 8x^3 - 8x = 8x(x+1)(x-1) \quad \text{and} \quad f''(x) = 24x^2 - 8 = 24\left(x+\sqrt{\tfrac{1}{3}}\right)\left(x-\sqrt{\tfrac{1}{3}}\right).$$

(a) (3 pts.) What are the critical point(s) of f and what is the value of f at each critical point?

The critical points of f *are* $x_1 = 0$, $x_2 = 1$, *and* $x_3 = -1$.

f(0) = 0 *and* f(-1) = f(-1) = 2.

(b) (3 pts.) Determine the open intervals where f is increasing or decreasing.

f *is decreasing on* $(-\infty,-1)\cup(0,1)$, *and increasing on* $(-1,0)\cup(1,\infty)$.

(c) (3 pts.) Determine the open intervals where f is concave up or concave down.

f *is concave down on* $(-1/\sqrt{3}, 1/\sqrt{3})$ *and concave up on* $(-\infty, -1/\sqrt{3})\cup(1/\sqrt{3}, \infty)$.

(d) (3 pts.) List any inflection points or state that there is none.

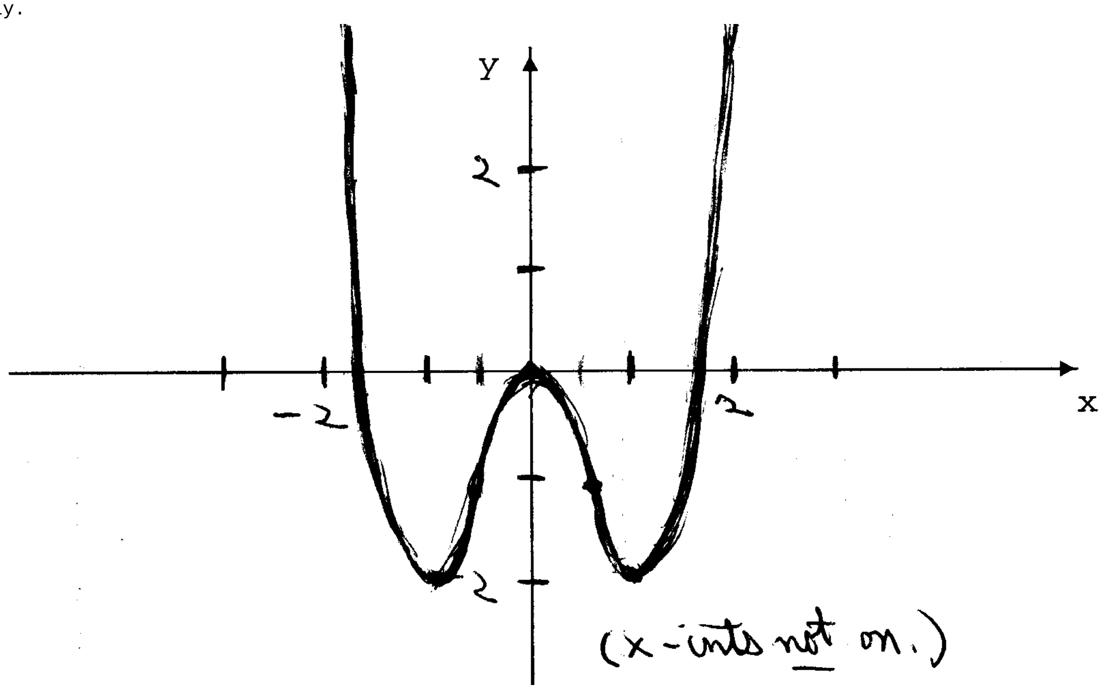
The inflection points of f *are* $(-1/\sqrt{3}, -10/9)$ *and* $(1/\sqrt{3}, -10/9)$.

(e) (3 pts.) Locate any x-intercepts of *f*, or state that there isn't any.

The x-intercepts may be read from the factorization of f:

$x = 0$, $x = -\sqrt{2}$, *and* $x = \sqrt{2}$.

(f) (5 pts.) Carefully sketch the graph of f below by plotting a few essential points and then connecting the dots appropriately.



Silly 10 point Bonus Problem: Show

$$\ln(x+1) \le x \ \ if \ \ x \ge 0.$$

[Say where your work is, for it won't fit here!]
Bottom of Page 2 of 5 might hold the goods.