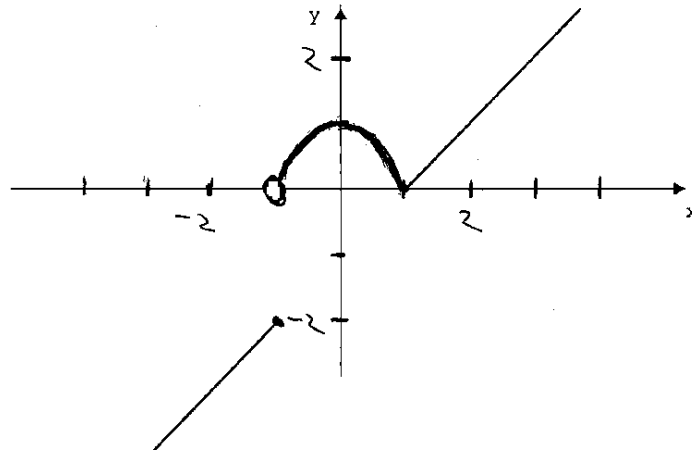


Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals", " \Rightarrow " denotes "implies", and " \Leftrightarrow " denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.

1. (20 pts.) Let

$$g(x) = \begin{cases} x-1, & \text{if } x \leq -1 \text{ or } 1 \leq x \\ 1-x^2, & \text{if } -1 < x < 1 \end{cases}$$

(a) (8 pts.) On the coordinate system below sketch the graph of the function g defined above. Label very carefully.



(b) (12 pts.) Evaluate each of the following easy limits concerning the function g above.

(i) $\lim_{x \rightarrow -\infty} g(x) = -\infty$ (ii) $\lim_{x \rightarrow -1^-} g(x) = -2$ (iii) $\lim_{x \rightarrow -1^+} g(x) = 0$

(iv) $\lim_{x \rightarrow 1} g(x) = 0$ (v) $\lim_{x \rightarrow +\infty} g(x) = \infty$

(vi) $\lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{(1 - x^2) - 1}{x} = \dots = 0$

2. (5 pts.) Express the following function in piecewise defined form without using absolute values:

$$f(x) = 5|x-2| - 4x = \begin{cases} x - 10, & \text{if } x \geq 2 \\ 10 - 9x, & \text{if } x < 2 \end{cases}$$

3. (25 pts.) For each of the following, find the limit if the limit exists. If the limit fails to exist, say so. Be as precise as possible here. [Work on the back of Page 2 of 4 if you run out of room here.]

$$(a) \quad \lim_{x \rightarrow 3} \frac{x^2 - 4}{x + 2} = 1$$

$$(b) \quad \lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2} = \lim_{x \rightarrow -2} (x - 2) = -4$$

$$(c) \quad \lim_{x \rightarrow 2^-} \frac{x + 2}{x^2 - 4} = \lim_{x \rightarrow 2^-} \frac{1}{x - 2} = -\infty$$

$$(d) \quad \lim_{x \rightarrow 1^-} \frac{5x - 5}{|x - 1|} = \lim_{x \rightarrow 1^-} \frac{5(x - 1)}{-(x - 1)} = -5$$

$$(e) \quad \lim_{x \rightarrow -\infty} \frac{\sqrt{25x^2 - 2}}{2x + 3} = \dots = \lim_{x \rightarrow -\infty} \frac{-\sqrt{25 - \frac{2}{x^2}}}{2 + \frac{3}{x}} = -\frac{5}{2}$$

Silly 10 point Bonus Problem: Provide an ϵ - δ proof that

$$\lim_{x \rightarrow 25} \sqrt{x} = 5.$$

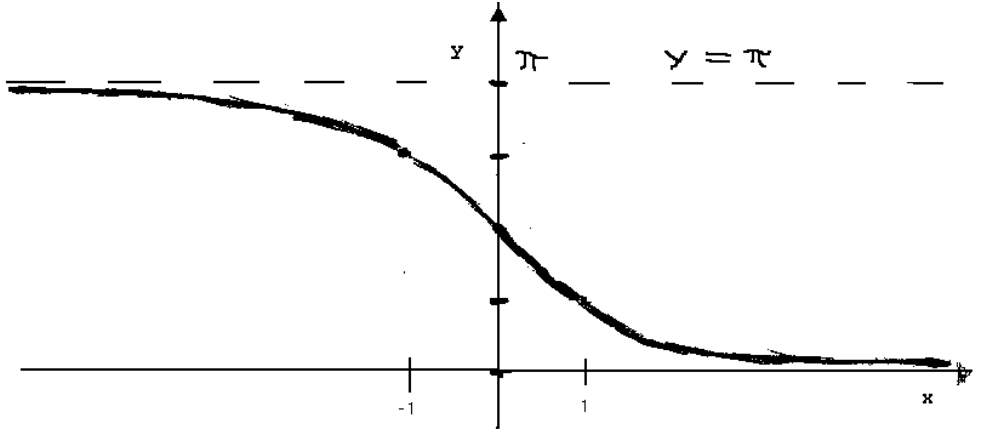
Proof: Let $\epsilon > 0$ be arbitrary. Then set $\delta = 5\epsilon$. Plainly $\delta > 0$. We shall now show that if $x \geq 0$ and $0 < |x - 25| < \delta$, then $|\sqrt{x} - 5| < \epsilon$. To this end, let $x \geq 0$ be any real number with $0 < |x - 25| < \delta$. Since

$$5 \leq \sqrt{x} + 5 \quad \Rightarrow \quad \frac{1}{\sqrt{x} + 5} \leq \frac{1}{5},$$

$$|\sqrt{x} - 5| = \left| \frac{x - 25}{\sqrt{x} + 5} \right| \leq \frac{|x - 25|}{5} < \frac{\delta}{5} = \epsilon.$$

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4. (7 pts.) Carefully sketch $f(x) = \cot^{-1}(x)$ on the coordinate system below. Label very carefully.



Then evaluate the following two limits:

$$\lim_{x \rightarrow \infty} \cot^{-1}(x) = 0$$

$$\lim_{x \rightarrow -\infty} \cot^{-1}(x) = \pi$$

5. (8 pts.) If $f(x) = 3x^2$ and $h \neq 0$, then simplifying as much as possible allows us to write

$$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h)^2 - 3x^2}{h} = \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} = \frac{h(6x + 3h)}{h} = 6x + 3h$$

6. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for

$$(*) \quad \lim_{x \rightarrow a} f(x) = L.$$

Suppose that f is a function that is defined everywhere in some open interval containing $x = a$, except possibly at $x = a$. We write $(*)$ if L is a number such that for each $\epsilon > 0$ we can find a $\delta > 0$, such that if x is in the domain of f and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

- (b) Using the $\epsilon - \delta$ definition of limit, provide a complete proof that

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6.$$

Proof: Let $\epsilon > 0$ be arbitrary. Set $\delta = \epsilon$. Then, if x is any number with $x \neq 3$ and $0 < |x - 3| < \delta$, then

$$\left| \frac{x^2 - 9}{x - 3} - 6 \right| = |(x + 3) - 6| = |x - 3| < \delta = \epsilon.$$

Since, for an arbitrary $\epsilon > 0$ we have found a number $\delta > 0$ so that $0 < |x - 3| < \delta$ implies that

$$\left| \frac{x^2 - 9}{x - 3} - 6 \right| < \epsilon$$

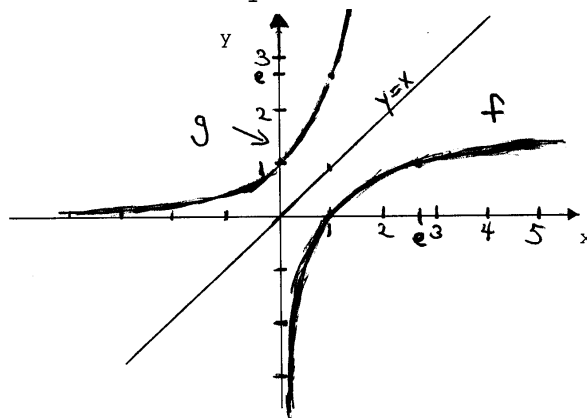
for $x \neq 3$, we are done.

7. (10 pts.) Evaluate each of the following thorny limits:

$$(a) \quad \lim_{x \rightarrow +\infty} (\sqrt{x^2 + 2ax} - x) = \dots = \lim_{x \rightarrow +\infty} \frac{2a}{\sqrt{1 + \frac{2a}{x}} + 1} = a$$

$$(b) \quad \lim_{x \rightarrow 0} \frac{\sqrt{x+16} - 4}{x} = \dots = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+16} + 4} = \frac{1}{8}$$

8. (15 pts.) (a) (5 pts.) Carefully sketch both $f(x) = \ln(x)$ and $g(x) = e^x$ on the coordinate system below. **Label very carefully.**



(b) (10 pts.) Evaluate each of the following limits.

$$\lim_{x \rightarrow 0^+} \ln(x) = -\infty$$

$$\lim_{x \rightarrow \infty} \ln(x) = \infty$$

$$\lim_{x \rightarrow -\infty} e^x = 0$$

$$\lim_{x \rightarrow \infty} e^x = \infty$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{4x} = \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{x}\right)^x\right]^4 = e^4$$

Silly 10 point Bonus Problem: Provide an $\varepsilon - \delta$ proof that

$$\lim_{x \rightarrow 25} \sqrt{x} = 5.$$

Say where your work is, for it won't fit here.

Look on the bottom of Page 2 of 4.