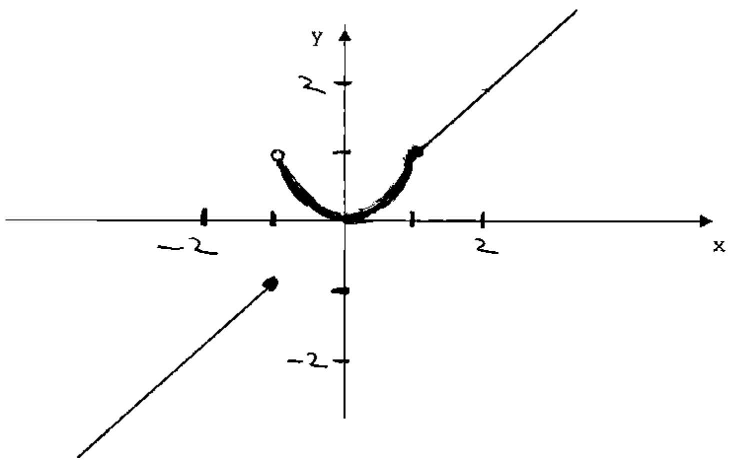
**Read Me First:** *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (20 pts.) Let

$$g(x) = \begin{cases} x & , \text{if } x \le -1 \text{ or } 1 \le x \\ x^2 & , \text{if } -1 < x < 1 \end{cases}$$

(a) (8 pts.) On the coordinate system below sketch the graph of the function *g* defined above. Label very carefully.




(b) (12 pts.) Evaluate each of the following easy limits concerning the function *g* above.

(i) $\lim_{x \to -\infty} g(x) = -\infty$  (ii) $\lim_{x \to -1^-} g(x) = -1$  (iii) $\lim_{x \to -1^+} g(x) = 1$

(iv) $\lim_{x \to 1} g(x) = 1$  (v) $\lim_{x \to +\infty} g(x) = \infty$

(vi) $\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 - 0}{x} = \ldots = 0$

2. (5 pts.) Express the following function in piecewise defined form without using absolute values:

$$f(x) = 4|x-2| - 5x = \begin{cases} -x - 8 & , \text{if } x \ge 2 \\ 8 - 9x & , \text{if } x < 2 \end{cases}$$

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3. (25 pts.) For each of the following, find the limit if the limit exists. If the limit fails to exist, say so. Be as precise as possible here. [Work on the back of Page 2 of 4 if you run out of room here.]

(a) $\lim\limits_{x \to -3} \dfrac{x^2-4}{x+2} = -5$

(b) $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x \to 2} (x+2) = 4$

(c) $\lim\limits_{x \to 2^-} \dfrac{x+2}{x^2-4} = \lim\limits_{x \to 2^-} \dfrac{1}{x-2} = \infty$

(d) $\lim\limits_{x \to 1^-} \dfrac{6x - 6}{|x - 1|} = \lim\limits_{x \to 1^-} \dfrac{6(x - 1)}{-(x - 1)} = -6$

(e) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{49x^2-2}}{2x+3} = \ldots = \lim\limits_{x \to -\infty} \dfrac{-\sqrt{49 - \dfrac{2}{x^2}}}{2 + \dfrac{3}{x}} = -\dfrac{7}{2}$

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Silly 10 point Bonus Problem: Provide an $\varepsilon$ - $\delta$ proof that

$$\lim_{x \to 16} \sqrt{x} = 4 .$$

Proof: Let $\varepsilon > 0$ be arbitrary. Then set $\delta = 4\varepsilon$. Plainly $\delta > 0$. We shall now show that if $x \geq 0$ and $0 < |x-16| < \delta$ , then $|\sqrt{x}-4| < \epsilon$ . To this end, let $x \geq 0$ be any real number with $0 < |x-16| < \delta$ . Since

$$4 \leq \sqrt{x}+4 \quad \Rightarrow \quad \frac{1}{\sqrt{x}+4} \leq \frac{1}{4} ,$$

$$|\sqrt{x}-4| = \left|\frac{x-16}{\sqrt{x}+4}\right| \leq \frac{|x-16|}{4} < \frac{\delta}{4} = \epsilon .$$

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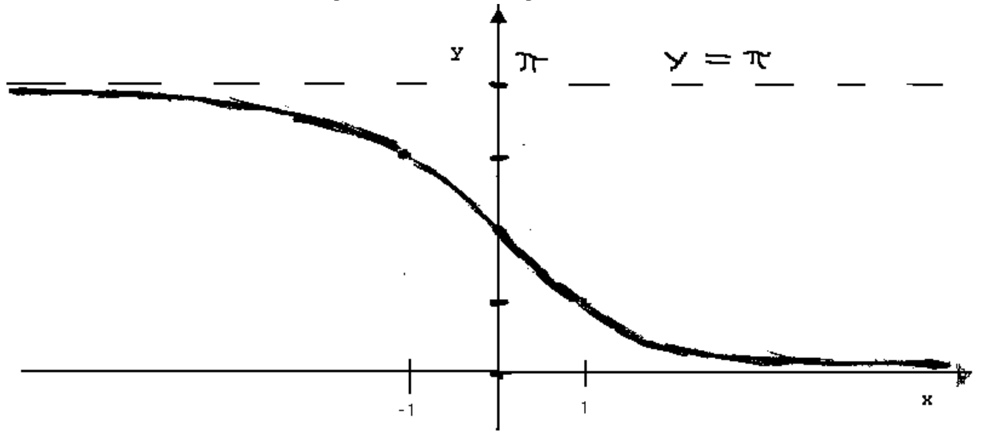
4. (7 pts.) Carefully sketch $f(x) = \cot^{-1}(x)$ on the coordinate system below. **Label very carefully**.




Then evaluate the following two limits:

$\lim_{x \to -\infty} \cot^{-1}(x) = \pi$ $\lim_{x \to \infty} \cot^{-1}(x) = 0$

5. (8 pts.) If $f(x) = 5x^2$ and $h \neq 0$, then simplifying as much as possible allows us to write

$$\frac{f(x+h)-f(x)}{h} = \frac{5(x+h)^2-5x^2}{h} = \frac{5x^2+10xh+5h^2-5x^2}{h} = \frac{h(10x+5h)}{h} = 10x+5h$$

6. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for

(*) $$\lim_{x \to a} f(x) = L.$$

Suppose that *f* is a function that is defined everywhere in some open interval containing *x* = *a*, except possibly at *x* = *a*. We write (*) if *L* is a number such that for each ε > 0 we can find a δ > 0, such that if *x* is in the domain of *f* and $0 < | x - a | < \delta$, then $| f(x) - L | < \varepsilon$.//

(b) Using the ε - δ definition of limit, provide a complete proof that

$$\lim_{x \to 4} \frac{x^2-16}{x-4} = 8.$$

Proof: Let ε > 0 be arbitrary. Set δ = ε. Then, if *x* is any number with $x \neq 4$ and $0 < | x - 4 | < \delta$, then

$$\left| \frac{x^2-16}{x-4} - 8 \right| = |(x+4) - 8| = |x-4| < \delta = \epsilon.$$

Since, for an arbitrary ε > 0 we have found a number δ > 0 so that $0 < | x - 4 | < \delta$ implies that

$$\left| \frac{x^2-16}{x-4} - 8 \right| < \epsilon$$

for $x \neq 4$, we are done.//

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7. (10 pts.) Evaluate each of the following thorny limits:

(a) $\lim_{x \to +\infty} \left( x - \sqrt{x^2 - 2bx} \right) = \ldots = \lim_{x \to +\infty} \dfrac{2b}{1 + \sqrt{1 - \dfrac{2b}{x}}} = b$

(b) $\lim_{x \to 0} \dfrac{\sqrt{x+4} - 2}{x} = \ldots = \lim_{x \to 0} \dfrac{1}{\sqrt{x+4} + 2} = \dfrac{1}{4}$

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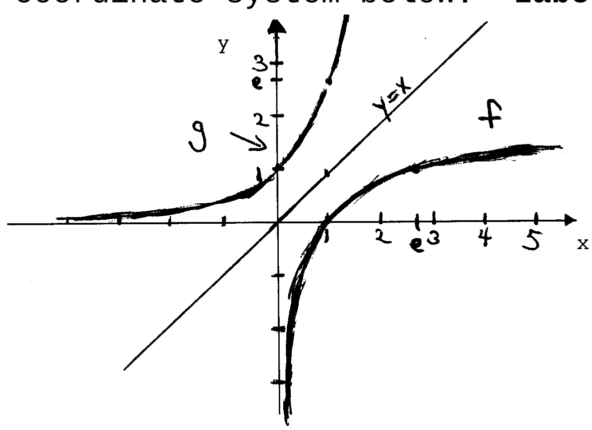
8. (15 pts.) (a) (5 pts.) Carefully sketch both f(x) = ln(x) and g(x) = $e^x$ on the coordinate system below. **Label very carefully**.




(b) (10 pts.) Evaluate each of the following limits.

$\lim_{x \to \infty} e^x = \infty$

$\lim_{x \to \infty} \ln(x) = \infty$

$\lim_{x \to -\infty} e^x = 0$

$\lim_{x \to 0^+} \ln(x) = -\infty$

$\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{5x} = \lim_{x \to \infty} \left[\left(1 + \dfrac{1}{x}\right)^{x}\right]^5 = e^5$

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Silly 10 point Bonus Problem: Provide an ε - δ proof that

$$\lim_{x \to 16} \sqrt{x} = 4.$$

Say where your work is, for it won't fit here.

*Look on the bottom of Page 2 of 4.*