Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Communicate. Show me all the magic on the page. Eschew obfuscation.*

1. (25 pts.) Compute the first derivatives of the following functions. You may use any of the rules of differentiation that are at your disposal. Do not attempt to simplify the algebra in your answers. You should do minor arithmetic, however, to have clean constants.

(a) $f(x) = 6x^4 - 12x^{-7} + 8\cot(x)$

$f'(x) = 24x^3 + 84x^{-8} - 8csc^2(x)$

(b) $g(x) = (2x^2 - 4x^{-1})csc(x)$

$g'(x) = (4x + 4x^{-2})csc(x) + (2x^2 - 4x^{-1})(-csc(x)\cot(x))$

(c) $h(t) = \dfrac{10t^5 + 1}{2 - \cos(t)}$

$$h'(t) = \frac{50t^4(2-\cos(t)) - (10t^5+1)\sin(t)}{(2-\cos(t))^2}$$

(d) $y = \tan^5(2\theta + 1)$

$$\frac{dy}{d\theta} = 5\tan^4(2\theta+1)\sec^2(2\theta+1)(2) = 10\tan^4(2\theta+1)\sec^2(2\theta+1)$$

(e) $L(z) = \sqrt{4z^8} + 4\sec(\frac{\pi}{6}) - 4\sin(\frac{z}{8})$

$$\frac{dL}{dz}(z) = \left(\frac{1}{2\sqrt{4z^8}}\right)(32z^7) - \frac{4}{8}\cos\left(\frac{z}{8}\right) = 8z^3 - \frac{1}{2}\cos\left(\frac{z}{8}\right)$$

10 point Bonus: Provide a rigorous ε-δ proof of the limit

$$\lim_{x\to 0} 2x^2\sin(\ln|x|) = 0$$

Let ε > 0 be arbitrary. Set $\delta = \sqrt{\frac{\epsilon}{2}}$. We'll now show this δ does the job. Suppose x is a real number such that $0 < |x| < \delta$. Then

$$\begin{aligned} |2x^2\sin(\ln|x|) - 0| &= |2x^2\sin(\ln|x|)| \\ &= |2x^2|\cdot|\sin(\ln|x|)| \\ &\le 2|x|^2 < 2\delta^2 = \epsilon. \end{aligned}$$

The key to the magic above is the inequality $|\sin(\ln|x|)| \le 1$ which you can see in a slightly different form in Problem 8. Another natural δ to use is δ = min(1, ε/2), where $x^2 \le |x|$.

2. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition of **continuity** of a function f(x) at a point x = a.

A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

(b) Is there a real number k, that will make the function $f(x)$ defined below continuous at $x = \pi/3$? Either find the value for k, or explain completely why there cannot be such a number k. Suppose

$$f(x) = \begin{cases} \dfrac{\cos(x) - (1/2)}{x - (\pi/3)} & , \ x \neq \pi/3 \\ k & , \ x = \pi/3 \end{cases}$$

From the definition of continuity at a point, in order for f to be continuous at $x = \pi/3$, it is necessary and sufficient for

$$k = f(\pi/3) = \lim_{x \to \pi/3} f(x) = \lim_{x \to 0} \frac{\cos(x) - (1/2)}{x - (\pi/3)}$$

$$= \cos'(\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2 .$$

We may also evaluate the key limit above correctly using the substitution $t = x - \pi/3$, a trigonometric identity for cosine, and the recognized derivatives for sine and cosine at t = 0.

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3. (5 pts.) Suppose that the line defined by $2x + 3y = 11$ is tangent to the graph of $y = f(x)$ at $x = 1$. Then $y = -\frac{2}{3}x + \frac{11}{3}$, so that

$f(1) = 3$ and $f'(1) = -\frac{2}{3}$

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4. (10 pts.) (a) Using complete sentences and appropriate notation, provide the precise mathematical definition for the derivative, $f'(x)$, of a function $f(x)$.

// The function f' defined by the equation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} .$$

is called the derivative of f with respect to x. The domain of f' consists of all x in the domain of f for which the limit above exists.//

(b) *Using only the definition of the derivative as a limit*, show all steps of the computation of $f'(x)$ when $f(x) = x^3$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2 \quad \textit{for every real number } x. \end{aligned}$$

5. (10 pts.) (a) Using complete sentences and appropriate notation, state the Intermediate-Value Theorem.

// If f is continuous on a closed interval $[a,b]$, and k is any number between $f(a)$ and $f(b)$, inclusive, then there is a number x_0 in the interval $[a,b]$ with $f(x_0) = k$.//

(b) Using the Intermediate-Value Theorem, show that the equation

$x^3 - x - 1 = 0$ has at least one solution in the interval [1,2].

// Let $f(x) = x^3 - x - 1$ on [1,2]. Observe that $f(1) = -1$, $f(2) = 5$, and $k = 0$ is a number between $f(1)$ and $f(2)$. Since f is a polynomial, f is continuous on the interval [1,2]. Consequently, the function f satisfies the hypotheses of the Intermediate-Value Theorem. Thus, we are entitled to invoke the magical conclusion that asserts that there is at least one number x_0 in (1,2) where $f(x_0) = 0$.//

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6. (10 pts.) A softball diamond is a square whose sides are 60 feet long. Suppose that a player running from first to second base has a speed of 25 feet per second at the instant when she is 10 feet from second base. At what rate is the player's distance from home plate changing at that instant?

Let $x(t)$ denote the distance from first base to the runner, and let $y(t)$ denote the distance from home plate to the runner. Then from the Pythagorean Theorem,

$$(*) \quad y^2(t) = (60)^2 + x^2(t)$$

while the runner is between first and second base. Consequently, during this time interval, by differentiating with respect to t, we see

$$2y(t)y'(t) = 2x(t)x'(t) \ .$$

At the moment in question, t_0, $x(t_0) = 50$ and $x'(t_0) = 25$. Using (*) and doing a bit of algebra reveals that

$$y'(t_0) = \frac{x(t_0)x'(t_0)}{y(t_0)} = \frac{(50)(25)}{(10)(61)^{1/2}} = \frac{125}{(61)^{1/2}} \ ft./\text{sec}.$$

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7. (10 pts.) Find the x-coordinates of all points on the graph of $y = 3 - x^2$ at which the tangent line passes through the point (2,0).

The first thing we need is an equation for the tangent line to the curve at an arbitrary point x_0 on the real line. Plainly, this is given by

$$y - (3 - x_0^2) = -2x_0(x - x_0) \ ,$$

or in slope-intercept form,

$$y = -(2x_0)x + x_0^2 + 3 \ ,$$

after routine algebra. The tangent line contains the point (2,0) precisely when

$$\begin{aligned} 0 &= -(2x_0)2 + x_0^2 + 3 \\ &= x_0^2 - 4x_0 + 3 \\ &= (x_0 - 1)(x_0 - 3). \end{aligned}$$

Thus, $x_0 = 1 \ \ or \ \ x_0 = 3$.

8. (5 pts.) (*) $\lim_{x \to 0} 2x^2 \sin(\ln|x|) = 0$

The limit above cannot be obtained using the arithmetic of limits. How does one legitimately see the truth of this limit easily??

The *Squeezing Theorem* is the appropriate tool. Observe that for $x \neq 0$,

$$-1 \le \sin(\ln|x|) \le 1 \quad \Rightarrow \quad -2x^2 \le 2x^2 \sin(\ln|x|) \le 2x^2 ,$$

and

$$\lim_{x \to 0} 2x^2 = 0 , \textit{ and } \lim_{x \to 0} (-2x^2) = 0 .$$

Consequently, the Squeezing Theorem implies the truth of (*) above.//

9. (5 pts.) Compute $f''(x)$ when $f(x) = x\tan(x)$.

$f'(x) = \tan(x) + x\sec^2(x)$

$f''(x) = \sec^2(x) + \sec^2(x) + x(2\sec(x)\sec(x)\tan(x))$

$= 2\sec^2(x) + 2x\sec^2(x)\tan(x)$

10. (10 pts.) Suppose that *f* and *g* are two differentiable functions with

$$f(2) = 3, \; f'(2) = -4, \textit{ and } f'(-4) = -8,$$

and

$$g(2) = -4, \; g'(2) = 5, \textit{ and } g'(3) = 7.$$

Using only the information above and appropriate differentiation rules, *either* compute the exact value of *h'*(2), when *h* is as defined in terms of *f* and *g* in each part *or* if there is essential data missing, say what it is.

(a) If $h(x) = 8f(x) + 25$, then $h'(2) = 8f'(2) = -32$

(b) If $h(x) = f(x)g(x)$, then $h'(2) = f'(2)g(2) + f(2)g'(2) = 31$

(c) If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(2) = \dfrac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{1}{16}$

(d) If $h(x) = f(g(x))$, then $h'(2) = f'(g(2))g'(2) = f'(-4)g'(2) = -40$

(e) If $h(x) = g(f(x))$, then $h'(2) = g'(f(2))f'(2) = g'(3)f'(2) = -28$

10 point Bonus: Provide a rigorous ε-δ proof of the limit in Problem 8, above. Say where your work is, for it won't fit here.

You'll find this on the bottom of Page 1 of 4.