
Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.

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1. (10 pts.) Write down each of the following derivatives. [2 pts/part.]

(a) $\dfrac{d[\tan^{-1}]}{dx}(x) = \dfrac{1}{1+x^2}$

(b) $\dfrac{d[\sin^{-1}]}{dx}(x) = \dfrac{1}{\sqrt{1-x^2}}$

(c) $\dfrac{d[\sec^{-1}]}{dx}(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$

(d) $\dfrac{d[\cos^{-1}]}{dx}(x) = \dfrac{-1}{\sqrt{1-x^2}}$

(e) $\dfrac{d[\cot^{-1}]}{dx}(x) = \dfrac{-1}{1+x^2}$

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2. (10 pts.) (a) Use logarithmic differentiation to find *dy/dx* when

$$y = x^{\tan(x)}$$

$$y = x^{\tan(x)} \quad \Rightarrow \quad \ln(y) = \ln(x)\cdot\tan(x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\tan(x) + \ln(x)\sec^2(x).$$

Thus,

$$\frac{dy}{dx} = \left[\frac{1}{x}\tan(x) + \ln(x)\sec^2(x)\right] x^{\tan(x)}.$$

(b) Find *dy/dx* when $\sin(xy) = y$.

We must use implicit differentiation here. First, pretend that *y* is a function of *x*. Then

$$\sin(xy) = y \quad \Rightarrow \quad \frac{d}{dx}(\sin(xy)) = \frac{dy}{dx}$$

$$\Rightarrow \quad \cos(xy)\left[y + x\frac{dy}{dx}\right] = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\cos(xy)}{1 - x\cos(xy)} \quad \textit{or equivalent.}$$

3. (10 pts.) (a) (6 pts.) Find formulas for Δy and the differential dy when $y = x^2$. Label your expressions correctly.

Evidentally $dy = (2x)dx$ is cheap thrills, but Δy is only slightly messier.

$$\begin{aligned}\Delta y &= (x+\Delta x)^2 - x^2 \\ &= [x^2+2x\Delta x+(\Delta x)^2] - [x^2] \\ &= (2x)\Delta x + (\Delta x)^2.\end{aligned}$$

(b) (4 pts.) Use an appropriate local linear approximation formula to estimate $\sqrt{24.9}$.

Let $f(x) = x^{1/2}$ and set $x_0 = 25$. Then the local linear approximation at x_0 is given by

$$\sqrt{x} = x^{1/2} \approx x_0^{1/2} + \frac{1}{2x_0^{1/2}}(x-x_0) = 5 + \frac{1}{10}(x - 25)$$

for all x. When $x = 24.9$,

$$\sqrt{24.9} = (24.9)^{1/2} \approx 5 + \frac{1}{10}(24.9 - 25) = 4.99 \ .$$

4. (10 pts.) (a) Let

$$f(x) = \frac{\exp(4-x^2)}{x} \quad for \quad x > 0 \ .$$

Show that f is one-to-one on $(0,\infty)$ using f'. [A little explanation is needed!]

Since

$$\begin{aligned}f'(x) &= \frac{-2x\exp(4-x^2)x - \exp(4-x^2)(1)}{x^2} \\ &= -\frac{(2x^2+1)\exp(4-x^2)}{x^2} < 0 \quad for \quad x > 0,\end{aligned}$$

f is strictly decreasing, and thus, one-to-one on $(0,\infty)$.

(b) Let $g(x) = f^{-1}(x)$, where f is from part (a) of this problem.

Compute $g'\left(\frac{1}{2}\right)$. Hint: $f(2) = ??$

$$g'\left(\tfrac{1}{2}\right) = (f^{-1})'\left(\tfrac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(2)} = -\frac{4}{9}.$$

5. (10 pts.) Fill in the blanks appropriately. [*DEFINITIONS!!!*]

(a) A function f has a relative minimum at x_0 if there is an open interval containing x_0 on which $f(x_0) \le f(x)$ for each x in both the interval and the domain of f.

(b) A function f is increasing on an interval (a, b) if $f(x_1) < f(x_2)$ whenever $a < x_1 < x_2 < b$.

(c) A function f is concave down on an interval (a, b) if the derivative of f is decreasing on (a, b).

(d) A function f is decreasing on an interval (a, b) if $f(x_1) > f(x_2)$ whenever $a < x_1 < x_2 < b$.

(e) A function f is concave up on an interval (a, b) if the derivative of f is increasing on (a, b).

6. (5 pts.) Suppose the side of a cube is measured to be 10 inches with a possible error of ± 1/10 inch. Estimate the error in the computed volume of the cube by using differentials. [You may leave your result as a fraction. You do not have to convert to decimal form.]

The volume of a cube is given by $V(x) = x^3$, where x is the length of a side in inches. Using differentials, we may approximate the error as follows:

$$\Delta V \approx dV = 3(x_0)^2 dx = 3(x_0)^2 \Delta x = (3)(10)^2(\pm 0.1) = \pm 30 \text{ cubic inches}$$

with $x_0 = 10$ and setting $dx = \Delta x = \pm 0.1$. This is not as bad as it looks when you consider that the computed volume is 1000 cubic inches.

7. (5 pts.) Find the following limit by interpreting the expression as an appropriate derivative. Complete the sentence outlined below appropriately.

(*) $$\lim_{x \to 2} \frac{10^x - 100}{x - 2} = \ln(10)10^2 = 100\ln(10)$$

since the limit (*) is actually $f'(a)$, where $f(x) = 10^x$ and $a = 2$.

8. (8 pts.) Assume f is continuous everywhere. If

$$f'(x) = (x-1)(x-2)^3,$$

find all the critical points of f and at each stationary point apply the second derivative test to determine relative extrema, if possible. If the second derivative test fails at a critical point, say as much and then apply the first derivative test to determine the true state of affairs.

We may read off of the derivative that the critical points are $x = 1$ and $x = 2$. Since

$$f''(x) = (x-2)^3 + 3(x-1)(x-2)^2,$$

$f''(1) = -1$ and $f''(2) = 0$. Thus, the second derivative test provides no information at $x = 2$, and implies that f has a relative maximum at $x = 1$.

When $1 < x < 2$, $f'(x) < 0$, and when $x > 2$, $f'(x) > 0$. The first derivative test now implies that f has a local minimum at $x = 2$.//

9. (12 pts.) Evaluate each of the following limits. If a limit fails to exist, say how as specifically as possible.

(a) $$\lim_{x\to 0} \frac{\sin^{-1}(3x)}{x} \overset{(L'H)}{=} \lim_{x\to 0} \frac{3}{\sqrt{1-(3x)^2}} = 3$$

This is also potentially a *recognized* derivative problem !

(b) $$\lim_{x\to 0^+} (\sin(x))^{4/\ln(x)} = \lim_{x\to 0^+} \exp\left(\frac{4\ln(\sin(x))}{\ln(x)}\right) = e^4$$ since

$$\lim_{x\to 0^-} \frac{4\ln(\sin(x))}{\ln(x)} \overset{(L'H+Alg.)}{=} \lim_{x\to 0^+} \frac{4x\cos(x)}{\sin(x)} = \lim_{x\to 0^+} 4\cos(x)\left(\frac{x}{\sin(x)}\right) = 4$$

and the exponential function is continuous.

(c) $$\lim_{x\to +\infty} (\ln(e^{x+5}+3) - x) = \lim_{x\to +\infty} \ln\left(\frac{e^{x+5}+3}{e^x}\right) = \lim_{x\to +\infty} \ln\left(e^5 + \frac{3}{e^x}\right) = \ln(e^5) = 5$$

Silly 10 point Bonus Problem: Show

$$(*) \quad \sqrt[3]{1+x} < 1 + \frac{1}{3}x \quad if \quad x > 0.$$

// Let $f(x) = 1 + (1/3)x - (1+x)^{1/3}$ for $x \geq 0$. Then f is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Plainly the truth of (*) above is equivalent to having $0 < x$ imply $0 < f(x)$. Now

$$f'(x) = \frac{1}{3} - \frac{1}{3}(1+x)^{-2/3} = \frac{((1+x)^{1/3}-1)((1+x)^{1/3}+1)}{3(1+x)^{2/3}}$$

for $x > 0$. Since $x > 0$ implies $(1 + x)^{1/3} > 1$, it follows that $f'(x) > 0$ when $x > 0$. Thus, f is increasing on $[0,\infty)$. Consequently, $0 < x$ implies $0 = f(0) < f(x)$.//

10. (20 pts.) When f is defined by

$$f(x) = 4x^3 - 3x^4 = -3x^3\left(x - \frac{4}{3}\right) \quad for\ x \in \mathbb{R},$$

$$f'(x) = 12x^2 - 12x^3 = -12x^2(x-1) \quad \text{and} \quad f''(x) = 24x - 36x^2 = -36x\left(x-\frac{2}{3}\right).$$

(a) (3 pts.) What are the critical point(s) of f and what is the value of f at each critical point?

The critical points of f are $x_1 = 0$, *and* $x_2 = 1$. $f(0) = 0$ *and* $f(1) = 1$.

(b) (3 pts.) Determine the open intervals where f is increasing or decreasing.

f is increasing on $(-\infty, 1)$, *and decreasing on* $(1, \infty)$.

(c) (3 pts.) Determine the open intervals where f is concave up or concave down.

f is concave up on $(0, 2/3)$ *and concave down on* $(-\infty, 0) \cup (2/3, \infty)$.

(d) (3 pts.) List any inflection points or state that there is none.

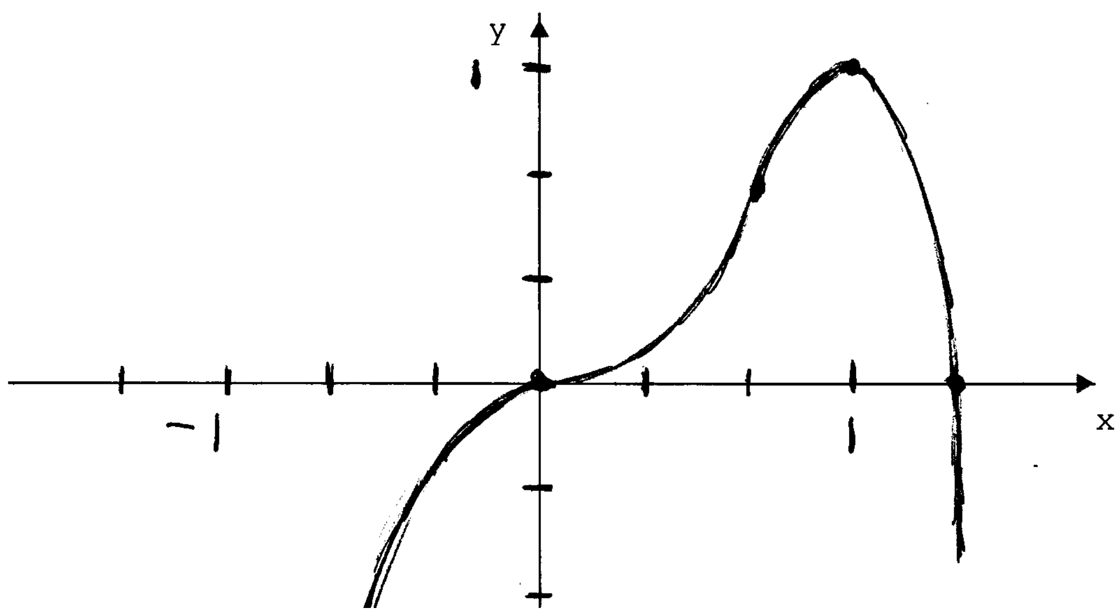
The inflection points of f are $(0, 0)$ *and* $(2/3, 16/27)$.

(e) (3 pts.) Locate any x-intercepts of f, or state that there isn't any.

The x-intercepts may be read from the factorization of f:

$$x_1 = 0 \text{ and } x_2 = 4/3.$$

(f) (5 pts.) Carefully sketch the graph of f below by plotting a few essential points and then connecting the dots appropriately.



Silly 10 point Bonus Problem: Show

$$\sqrt[3]{1+x} < 1 + \frac{1}{3}x \quad if \quad x > 0.$$

[Say where your work is, for it won't fit here!]

This may be found briefly on the bottom of Page 4 of 5.