READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds.*

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1. (5 pts.) Complete the equation to write the following sum using sigma notation:

$$\frac{x^1}{2} - \frac{x^2}{4} + \frac{x^3}{6} - \frac{x^4}{8} + \ldots - \frac{x^{22}}{44} = \sum_{i=1}^{22} \frac{(-1)^{i+1}x^i}{2i}$$

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2. (5 pts.) Find the average value of f(x) = sin(x) over the interval [π/3,π].

$$\begin{aligned} f_{AVE} = \frac{1}{\pi-(\pi/3)}\int_{\pi/3}^{\pi} \sin(x)\, dx &= \frac{3}{2\pi}(-\cos(x))\Big|_{\pi/3}^{\pi} \\ &= \frac{3}{2\pi}(-\cos(\pi)+\cos(\pi/3)) \\ &= \frac{3}{2\pi}\cdot\frac{3}{2} \\ &= \frac{9}{4\pi}. \end{aligned}$$

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3. (10 pts.) Differentiate the following functions:

(a) $g(x) = \int_x^0 \frac{1}{(1-t^2)^{1/2}}\, dt$ [Note: The domain of g is (-1,1).]

$$g'(x) = \frac{-1}{(1-x^2)^{1/2}}$$

(b) $f(x) = \int_1^{e^x} \frac{1}{1+t^2}\, dt$

$$f'(x) = \frac{e^x}{1+e^{2x}}$$

Make sure you label your derivatives correctly.

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4. (5 pts.) Using appropriate properties of the definite integral and suitable area formulas from geometry, evaluate the following definite integral. [Roughly sketching a couple of simple graphs might help. First use linearity, though.]

$$\int_0^3 2(9 - x^2)^{1/2} - x\,dx \quad = \quad 2\int_0^3 (9 - x^2)^{1/2}\,dx - \int_0^3 x\,dx$$

$$= \quad 9\pi/2 \quad - \quad 9/2$$

since the numerical value of the first integral is simply the area of a quarter circle with a radius of 3 and the value of the second integral is the area of a right triangle with two legs of length 3.

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5. (5 pts.) State the Fundamental Theorem of Calculus.
 Suppose that f is continuous on the closed interval [a,b].

Part 1: If the function g is defined on [a,b] by

$$g(x) = \int_a^x f(t)\,dt,$$

then g is an antiderivative of f. That is, $g'(x) = f(x)$ for each x in [a,b]. [This, really, is the punch line!!]

Part 2: If G is any antiderivative of f on [a,b], then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

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6. (5 pts.) Express the solution to the following initial value problem by using a definite integral with respect to the variable t,

$$\frac{dy}{dx} = (1 + e^x)^{1/2}\,, \quad y(1) = 10.$$

To do this, fill in the right side of the equation below correctly.

$$y(x) = \quad 10 + \int_1^x (1+e^t)^{1/2}\,dt$$

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7. (5 pts.) Using only the second comparison property of integrals, give both a lower and an upper bound on the true numerical value of the integral below.

$$I \quad = \quad \int_0^{\pi/6} \cos^2(x)\,dx$$

Let $f(x) = \cos^2(x)$. Then $f'(x) = -2\cos(x)\sin(x) < 0$ when x is in the interval $(0,\pi/6)$. So f is decreasing on $[0,\pi/6]$. Thus, $3/4 = f(\pi/6) \le f(x) \le f(0) = 1$ when $0 \le x \le \pi/6$. From the 2nd comparison property of integrals, it follows that we have $\pi/8 = (3/4)(\pi/6) \le I \le (1)(\pi/6) = \pi/6$.

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8. (10 pts.) Assume that $[x_{i-1},x_i]$ denotes the *ith* subinterval of a partition of the interval [0,1] into n subintervals, all with the same length $\Delta x = 1/n$.

(a) Write the value of the following limit as a definite integral.

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} e^{2x_i}\Delta x$$

$$L \quad = \quad \int_0^1 e^{2x}\,dx$$

(b) By evaluating the integral you obtained in part (a) above using the Fundamental Theorem of Calculus, give the exact numerical value of the limit, L.

$$L \quad = \quad \int_0^1 e^{2x}\,dx = \left(\frac{1}{2}e^{2x}\right)\Big|_0^1 = \frac{1}{2}(e^2 - 1).$$

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9. (10 pts.) Reveal all the details of evaluating the given integral by computing

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where the sum is assumed to have originated from a regular partition of the given interval of integration. Here, you are to actually compute the Riemann sum in closed form and then evaluate the limit.

Since $\Delta x = 2/n$, $x_i = 2i/n$ for i = 0, 1, ... , n are the end points of the intervals of the general regular partition. So

$$\begin{aligned}
\int_0^2 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \Delta x \\
&= \lim_{n\to\infty} \left(\frac{8}{n^3}\right)\sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \left(\frac{8}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \\
&= \frac{8}{3}.
\end{aligned}$$

10. (5 pts.) Give an example of a bounded function defined on the closed interval [0,1] that is not Riemann integrable.

Here is an obnoxious old friend that does the job.

$$g(x) = \begin{cases} 0 & , \quad \text{for } x \text{ a rational number in } [0,1] \\ 1 & , \quad \text{for } x \text{ an irrational number in } [0,1] \end{cases}$$

5.4: Problem 55.

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11. (5 pts.) Are there any problems with the string of equations used in the computation

$$\int_0^{\pi/4} \tan^2(x)\sec^2(x)dx = \int_0^{\pi/4} u^2 du = \left(\frac{1}{3}u^3\right)\Big|_0^{\pi/4} = \left(\frac{1}{3}\tan^3(x)\right)\Big|_0^{\pi/4} = \frac{1}{3}$$

when we use the substitution u = tan(x) so that du = $\sec^2(x)dx$?? Explain briefly.

Why yes, indeed, there is a problem here. The first and third equations aren't true. The first equation is false as a result of a failure to apply the substitution theorem correctly by changing the limits of integration. The third equation is made false by substituting back, replacing u with tan(x). Go ahead and compute the two differences and take note of how different the results are.

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12. (10 pts.) (a) Locate the critical points of the function g defined on the interval $[0,4\pi]$ by means of the equation

$$g(x) = \int_0^x t\sin(t)\, dt.$$

[Note: The critical points are actually in the open interval $(0,4\pi)$.]

From the Fundamental Theorem of Calculus, it follows that $g'(x) = x\cdot\sin(x)$ for $0 < x < 4\pi$. It follows that $g'(x) = 0$ precisely where $\sin(x) = 0$ in $(0,4\pi)$. Thus, the critical points of g are $x = \pi$, $x = 2\pi$, and $x = 3\pi$.

(b) Determine the open intervals in $(0,4\pi)$ where g is increasing or decreasing.

Plainly, if $x > 0$, the sign of $g'(x)$ is determined by the sign of $\sin(x)$ on $(0,4\pi)$. Thus, $g'(x) > 0$ when $0 < x < \pi$ or $2\pi < x < 3\pi$, and $g'(x) < 0$ when $\pi < x < 2\pi$ or $3\pi < x < 4\pi$. It follows that g is increasing on the set $(0,\pi)\cup(2\pi,3\pi)$ and g is decreasing on the set $(\pi,2\pi)\cup(3\pi,4\pi)$.

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13. (10 pts.) Evaluate each of the following sums in closed form.

(a) $$\sum_{i=0}^{199}\left(\frac{1}{2}\right)^i = \frac{1-(1/2)^{(199+1)}}{1-(1/2)} = 2[\ 1-(1/2)^{200}\ ]$$

(b) $$\sum_{i=1}^{200}(2i+1) = 2\sum_{i=1}^{200} i + \sum_{i=1}^{200} 1 = \frac{2(200)(201)}{2} + 200$$
$$= (200)(202) = 40400$$

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14. (10 pts.) Evaluate the following definite integral.

$$\int_0^2 |\ x^2-1\ |\ dx = \int_0^1 |\ x^2-1\ |\ dx + \int_1^2 |\ x^2-1\ |\ dx$$
$$= \int_0^1 -(x^2-1)\ dx + \int_1^2 x^2-1\ dx$$
$$= (x-(x^3/3))\Big|_0^1 + ((x^3/3)-x)\Big|_1^2$$
$$= \ldots = 2.$$

Silly 10 Point Bonus: Theorem 1 of Section 5.6 is called the Average Value Theorem, and its statement follows:

If f is continuous on [a,b], then

(*) $$f(\bar{x}) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

for some $\bar{x}$ in [a,b].

Provide a proof of this by briefly explaining how the conclusion follows from two very important properties of continuous functions via the integral comparison properties.
[Several sentences are needed. Work on the back of Page 4 of 5.]

Proof: Pretend f is continuous on [a,b]. The Extreme Value Theorem implies that there are numbers c and d in [a,b] such that

(**) $$m = f(c) \le f(x) \le f(d) = M$$

is true for each real number x in [a,b]. Now f being continuous implies that f is integrable. The integral comparison properties and (**) above now imply that

$$m(b-a) \le \int_a^b f(x)\ dx \le M(b-a).$$

By multiplying through by the reciprocal of (b-a), it follows that

$$m \le \frac{1}{b-a}\int_a^b f(x)\ dx \le M.$$

Since the average value is a number in the closed interval containing the maximum and minimum of f on [a,b], the Intermediate Value Theorem implies that there is a number $\bar{x}$ in [a,b] so that (*) is true.//