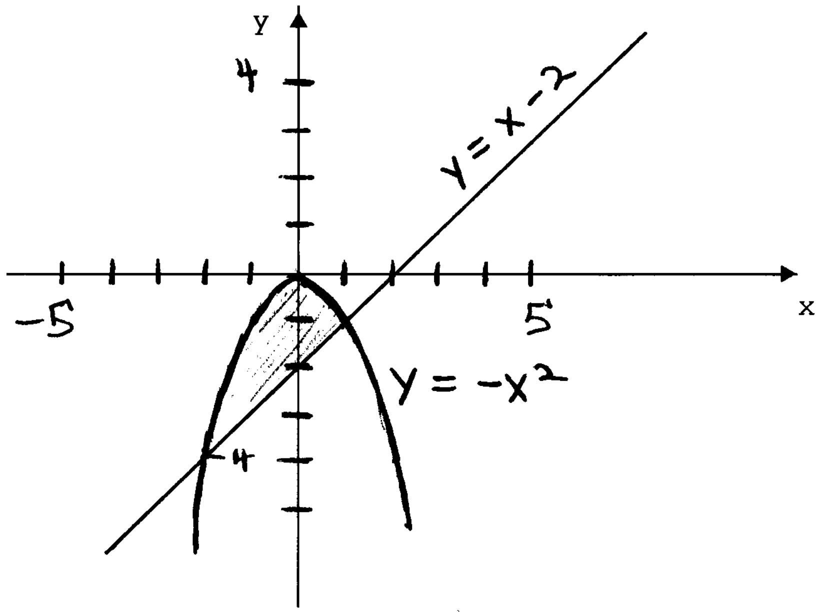
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (15 pts.) (a) Sketch very carefully the bounded region bounded by the curves $y = -x^2$ and $y = x - 2$ on the coordinate system provided. Label very carefully. (b) Simply write the definite integral, dx, that yields the area of the region. (c) Evaluate the definite integral of part (b) using the Fundamental Theorem of Calculus.

(a)



(b) Area = $\int_{-2}^{1} -x^2 - (x - 2)\, dx = \int_{-2}^{1} 2 - x - x^2\, dx$

(c) $= \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-2}^{1} = \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = 5 - \frac{1}{2} = \frac{9}{2}.$

2. (5 pts.) Give the definition of the function ln(x) in terms of a definite integral and give its domain and range. Label correctly. (**Hint**: Complete the sentence,"ln(x) = ...")

$\ln(x) = \int_1^x \frac{1}{t}\, dt$ *for* $x > 0$. The range of the natural logarithm is the whole real line, $\mathbb{R} = (-\infty, \infty)$.

3. (10 pts.) Compute (a) the net distance, and (b) the total distance, traveled between time t = 0 and time t = π by a particle moving with the velocity function v = cos(t).

(a)

$$\text{Net\_Distance} = \int_0^\pi v(t)dt = \int_0^\pi \cos(t)dt = \sin(\pi) - \sin(0) = 0 \ .$$

(b)

$$\begin{aligned}\text{Total\_Distance} = \int_0^\pi |v(t)|dt &= \int_0^\pi |\cos(t)|dt \\ &= \int_0^{\pi/2} |\cos(t)|dt + \int_{\pi/2}^\pi |\cos(t)|dt \\ &= \int_0^{\pi/2} \cos(t)dt + \int_{\pi/2}^\pi (-\cos(t))dt \\ &= \ldots = 2.\end{aligned}$$

4. (10 pts.) Obtain the arc length along the curve defined by the equation $y = (4 - x^2)^{1/2}$ from $x = -(2)^{1/2}$ to $x = (3)^{1/2}$.

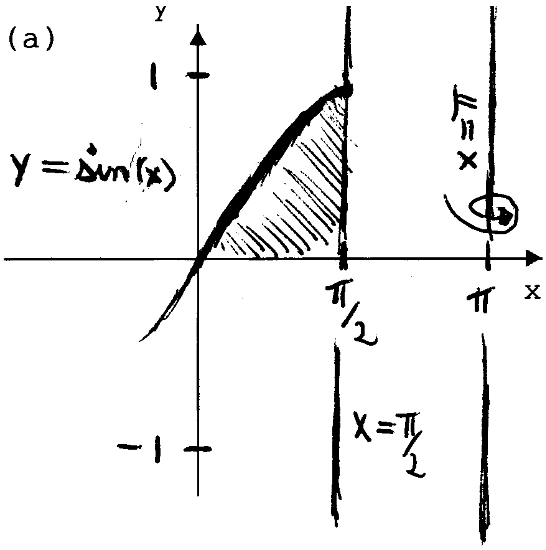
Since $\frac{dy}{dx} = \frac{1}{2}(4-x^2)^{-1/2}(-2x) = \frac{-x}{(4-x^2)^{1/2}}$, it follows that

$$\begin{aligned} L &= \int_{-2^{1/2}}^{3^{1/2}} \left(1+\left(\frac{-x}{(4-x^2)^{1/2}}\right)^2\right)^{1/2} dx \\ &= \int_{-2^{1/2}}^{3^{1/2}} \frac{1}{\left(1-\left(\frac{x}{2}\right)^2\right)^{1/2}}\, dx \\ &= \int_{-2^{1/2}/2}^{3^{1/2}/2} \frac{2}{(1-u^2)^{1/2}} du \\ &= 2\sin^{-1}(3^{1/2}/2) - 2\sin^{-1}(-2^{1/2}/2) = 2\left[\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)\right] = \frac{7\pi}{6}. \end{aligned}$$

Obviously, we have used the substitution, u = x/2 above.

5. (10 pts.) (a) Sketch the region in the 1st quadrant enclosed by the curves defined by $y = \sin x$, $y = 0$, and $x = \pi/2$. Suppose the region is revolved around the line defined by $x = \pi$. (b) Using the method of cylindrical shells, write down the definite integral used to compute the volume of the solid of revolution formed. **Don't evaluate the integral.** (c) Using the method of slicing [disks/washers here], write down the definite integral used to compute the same volume as in part (b). **Don't evaluate the integral.**

(a)



(b)

$$V = \int_0^{\pi/2} 2\pi(\pi - x)\sin(x)\,dx$$

(c)

$$V = \pi\int_0^1 (\pi - \sin^{-1}(y))^2 - (\pi - \pi/2)^2\,dy$$
$$= \pi\int_0^1 (\pi - \sin^{-1}(y))^2 - \frac{\pi^2}{4}\,dy$$

6. (10 pts.) Consider the definite integral below. (a) Write down the sum, S_4, used to approximate the value of the integral below if Simpson's Rule is used with $n = 4$. **Do not attempt to evaluate the sum. Be very careful.** (b) Write down the sum, T_4, used to approximate the value of the integral below if Trapezoid Rule is used with $n = 4$. **Do not attempt to evaluate the sum. Be very careful.**

$$\int_1^2 x^{1/3}\,dx$$

Since $\Delta x = 1/4$, the endpoints of the regular partition we are using are $x_0 = 1$, $x_1 = 5/4$, $x_2 = 6/4$, $x_3 = 7/4$, and $x_4 = 2$.

(a) $S_4 = \frac{1}{12}\cdot(1^{1/3} + 4(5/4)^{1/3} + 2(6/4)^{1/3} + 4(7/4)^{1/3} + 2^{1/3})$

(b) $T_4 = \frac{1}{8}\cdot(1^{1/3} + 2(5/4)^{1/3} + 2(6/4)^{1/3} + 2(7/4)^{1/3} + 2^{1/3})$

7. (20 pts.) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.
[**5 pts./part**]
(a)

$$\int (2t+1)\cos(t)\,dt = (2t+1)\sin(t) - \int 2\sin(t)\,dt$$
$$= (2t+1)\sin(t) + 2\cos(t) + C$$

by integrating by parts with u = 2t+1 and dv = cos(t)dt.

(b)

$$\int_0^{(\pi/3)^{1/2}} (4x)\sin(x^2)\,dx = \int_0^{\pi/3} 2\sin(u)\,du$$
$$= (-2\cos(u))\Big|_0^{\pi/3}$$
$$= -2\cos(\pi/3)-(-2\cos(0)) = 1$$

by using the u-substitution u = x^2 and the usual trigonometric treats.

(c)

$$\int_1^e \frac{1}{x[1 + (\ln(x))^2]}\,dx = \int_0^1 \frac{1}{1 + u^2}\,du$$
$$= \tan^{-1}(u)\Big|_0^1$$
$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

by using the u-substitution u = ln(x) and by spotting the lurking arctangent Michael.

(d)

$$\int (t+1)e^{2t}\,dt = (t+1)\frac{e^{2t}}{2} - \int \frac{e^{2t}}{2}\,dt$$
$$= (t+1)\frac{e^{2t}}{2} - \frac{e^{2t}}{4} + C$$

by integrating by parts with u = t+1 and dv = e^{2t} dt.

8. (10 pts.) Suppose that $n \geq 2$ is a positive integer. Show in detail how to derive the following reduction formula:

$$\int \sin^n(x)\,dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

First, factor $\sin^n(x)$. Then do an integration by parts after choosing $u = \sin^{n-1}(x)$ and $dv = \sin(x)dx$. Then

$$\begin{aligned}\int \sin^n(x)dx &= \int \sin^{n-1}(x)\sin(x)dx \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)dx\end{aligned}$$

since $du = (n-1)\sin^{n-2}(x)\cos(x)dx$ and $v = -\cos(x)$.

By replacing $\cos^2(x)$ with $1 - \sin^2(x)$, doing the obvious algebra, and using the linearity of the integral, you may now produce

$$\int \sin^n(x)dx = -\sin^{n-1}(x)\cos(x) - (n-1)\int \sin^n(x)dx + (n-1)\int \sin^{n-2}(x)dx$$

After adding $(n-1)\int \sin^n(x)dx$ to both sides of the equation above, and simplifying the left side algebraically, we have

$$n\int \sin^n(x)dx = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)dx$$

Multiplying by n^{-1} on both sides of this equation finishes the incantation.

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9. (10 pts.) Differentiate each of the following functions.

(a) $f(x) = \sec^{-1}(x)$ $\quad f'(x) = \dfrac{1}{|x|(x^2-1)^{1/2}}$

(b) $f(x) = \cot^{-1}(x)$ $\quad f'(x) = -\dfrac{1}{1+x^2}$

(c) $f(x) = 20^x$ $\quad f'(x) = \ln(20)20^x$

(d) $f(x) = \sin^{-1}(x)$ $\quad f'(x) = \dfrac{1}{(1-x^2)^{1/2}}$

(e) $f(x) = \log_\pi(x)$ $\quad f'(x) = \dfrac{1}{x\cdot\ln(\pi)} = \log_\pi(e)\cdot\dfrac{1}{x}$

Silly 10 Point Bonus: Consider the following equation:

$$\tan^{-1}\left(\frac{x}{(1-x^2)^{1/2}}\right) = \int_0^x \frac{1}{(1-t^2)^{1/2}}\,dt$$

Is this equation true for each x in the open interval (-1,1)?? On the back of Page 4 of 5, either provide me proof that the equation is true for every x in (-1,1), or show me that there is some number x_0 in the interval where the equation fails to be true.[Note: An answer for this may be found in a separate document along the Test 2 row. It is "c2-t2-b.pdf".]