READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (30 pts.) Here are five trivial trigonometric integrals to evaluate. [6 pts./part]

(a) $\int \tan^2(x)dx = \int \sec^2(x) - 1dx = \tan(x) - x + C$

(b)
$$\begin{aligned}\int \cot(4x)dx &= \frac{1}{4}\int \cot(u)du \\ &= -\frac{1}{4}\ln|csc(u)| + C \\ &= -\frac{1}{4}\ln|csc(4x)| + C \\ &\textit{or} \\ &= \frac{1}{4}\ln|\sin(4x)| + C \ \ \textit{using } u = 4x.\end{aligned}$$

(c)
$$\begin{aligned}\int \frac{\sin^2(t)}{\cos(t)}dt &= \int \frac{1 - \cos^2(t)}{\cos(t)}dt \\ &= \int \sec(t) - \cos(t)dt \\ &= \ln|\sec(t) + \tan(t)| - \sin(t) + C\end{aligned}$$

(d)
$$\begin{aligned}\int \cos(x)\cos(4x)dx &= \int \frac{\cos(5x) + \cos(-3x)}{2}dx \\ &= \frac{1}{2}\int \cos(5x) + \cos(3x)dx \\ &= \frac{\sin(5x)}{10} + \frac{\sin(3x)}{6} + C\end{aligned}$$

(e)
$$\begin{aligned}\int \tan(t)\sec^4(t)dt &= \int \sec^3(t)\sec(t)\tan(t)dt \\ &= \int u^3du = \frac{u^4}{4} + C \\ &= \frac{\sec^4(t)}{4} + C \ \ \textit{using } u = \sec(t),\end{aligned}$$

or

$$\begin{aligned}\int \tan(t)\sec^4(t)dt &= \int \sec^2(t)\sec^2(t)\tan(t)dt \\ &= \int (\tan^2(t)+1)\tan(t)\sec^2(t)dt \\ &= \int u^3 + u\,du = \frac{u^4}{4} + \frac{u^2}{2} + C \\ &= \frac{\tan^4(t)}{4} + \frac{\tan^2(t)}{2} + C \ \ \textit{using } u = \tan(t).\end{aligned}$$

2. (20 pts.) Evaluate each of the following antiderivatives [5 pts./part]

(a) If x > 1, then

$$\int \sec^{-1}(x)dx = x\sec^{-1}(x) - \int \frac{x}{|x|(x^2-1)^{1/2}}dx$$
$$= x\sec^{-1}(x) - \int \frac{1}{(x^2-1)^{1/2)}}dx$$
$$= x\sec^{-1}(x) - \ln|x + (x^2-1)^{1/2}| + C$$

after integrating by parts with $u = \sec^{-1}(x)$ *and* $dv = 1\,dx$

since

$$\int \frac{1}{(x^2-1)^{1/2}}dx = \int \frac{\sec(\theta)\tan(\theta)}{\tan(\theta)}d\theta$$
$$= \int \sec(\theta)\,d\theta$$
$$= \ln|\sec(\theta) + \tan(\theta)| + C$$
$$= \ln|x + (x^2-1)^{1/2}| + C$$

by using the trigonometric substitution $x = \sec(\theta)$.

(b)

$$\int \frac{1}{x^2-2x+26}dx = \int \frac{1}{(x-1)^2+25}dx$$
$$= \frac{1}{25}\int \frac{1}{1+(\frac{x-1}{5})^2}dx$$
$$= \frac{1}{5}\int \frac{1}{1+u^2}du$$
$$= \frac{1}{5}\tan^{-1}(u) + C$$
$$= \frac{1}{5}\tan^{-1}\left(\frac{x-1}{5}\right) + C$$

using the u-substitution $u = \frac{x-1}{5}$.

(c)

$$\int (4 - t^2)^{1/2}dt = 2\int (1 - (\frac{t}{2})^2)^{1/2}dt$$
$$= 2\int \cos(\theta)\,2\cos(\theta)\,d\theta$$
$$= 4\int \frac{1+\cos(2\theta)}{2}d\theta$$
$$= 2\theta + \sin(2\theta) + C$$
$$= 2\sin^{-1}(\frac{t}{2}) + \frac{t(4 - t^2)^{1/2}}{2} + C$$

using the trigonometric substitution $\sin(\theta) = \frac{t}{2}$.

(d)

$$\int \frac{x^4}{x^3+x}dx = \int x - \frac{x}{x^2+1}dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C,$$

after getting rid of a common factor and doing division.

3. (10 pts.) (a) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc.

$$\frac{x^2+4}{x(x-1)^3(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{Ex+F}{x^2+1}$$

(b) Now obtain the indefinite integral of the rational function of part (a) in terms of the constants A, B, C, etc. using the partial fraction decomposition. Do not attempt to obtain the numerical values of the constants.

$$\int \frac{x^2+4}{x(x-1)^3(x^2+1)}dx = A\ln|x| + B\ln|x-1| - C(x-1)^{-1} - \frac{D}{2}(x-1)^{-2} + E\ln|x^2+1|^{1/2} + F\tan^{-1}(x) + K$$

Here, of course, we have used the linearity of the integral and have done a few u-substitutions in our head. [Yes, this is this easy. Just don’t get spooked by the silly letters.]

4. (5 pts.) Evaluate the following, quite proper, definite integral:

$$\int_{-1}^{1} \frac{1}{(1+x^2)^{1/2}}dx = \int_{-\pi/4}^{\pi/4} \sec(\theta)d\theta$$

$$= (\ln|\sec(\theta)+\tan(\theta)|)\Big|_{-\pi/4}^{\pi/4}$$

$$= \ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1)$$

or

$$= \ln(3 + 2\sqrt{2}).$$

Plainly, we have used the trigonometric substitution $x = \tan(\theta)$ right away. So $dx = \sec^2(\theta)\,d\theta$, and the values of θ for the limits of integration obviously have come from $\theta = \tan^{-1}(x)$.

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5. (5 pts.) Find a pattern in the sequence with given terms a_1, a_2, a_3, a_4 , and assuming that it continues as indicated, write a formula for the general term a_n of the sequence.

-1/2, +3/5, -5/8, +7/11, ...

Clearly, $a_n = \dfrac{(-1)^n(2n-1)}{2+3(n-1)} = \dfrac{(-1)^n(2n-1)}{3n-1}$ *for* $n \geq 1$.

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6. (5 pts.) Assume that the sequence $\{x_n\}$ is defined recursively by the formula

$$x_{n+1} = (5 + x_n)^{1/2} \text{ if } n \geq 1.$$

If $x_1 = 1$ and if $L = \lim_{n\to\infty} x_n$ exists, what is the exact value of L?? Since

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (5 + x_n)^{1/2} = (5 + \lim_{n\to\infty} x_n)^{1/2} = (5 + L)^{1/2},$$

the limit L is a solution to the quadratic equation

$$L^2 - L - 5 = 0.$$

Thus,using the quadratic formula, it follows that

$$L = \frac{1 + (21)^{1/2}}{2} \text{ or } L = \frac{1 - (21)^{1/2}}{2}.$$

Since the sequence is non-negative, the second root is not a possible value for the limit.

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7. (10 pts.) Determine whether the sequence $\{a_n\}$ converges, and find its limit if it does.

(a)

$$\begin{aligned} a_n &= \int_1^n \frac{1}{x^2+x}dx = \int_1^n \frac{1}{x(x+1)}dx \\ &= \int_1^n \frac{1}{x} - \frac{1}{x+1}dx \\ &= (\ln(x) - \ln(x+1))\Big|_1^n \\ &= \left(\ln\left(\frac{x}{x+1}\right)\right)\Big|_1^n = \ln\left(\frac{n}{n+1}\right) - \ln\left(\frac{1}{2}\right) \end{aligned}$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\ln\left(\frac{n}{n+1}\right) - \ln\left(\frac{1}{2}\right)\right] = -\ln\left(\frac{1}{2}\right) = \ln(2).$$

(b) $a_n = n \cdot \sin(\frac{2\pi}{n})$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2\pi \frac{\sin(\frac{2\pi}{n})}{\frac{2\pi}{n}} = 2\pi.$$

[You could use L'Hopital's rule but it is really not needed.]

8. (15 pts.) Evaluate each of the following integrals. Look before you leap.

(a)

$$\int_0^{\infty} e^{-5x}\,dx = \lim_{t\to\infty}\int_0^t e^{-5x}\,dx$$
$$= \lim_{t\to\infty}\left(-\frac{1}{5}e^{-5x}\right)\Big|_0^t$$
$$= \lim_{t\to\infty}\left(\frac{1}{5} - \frac{1}{5}e^{-5t}\right)$$
$$= \frac{1}{5}.$$

(b)

$$\int_0^{10}(10-x)^{-1}\,dx = \lim_{t\to 10^-}\int_0^t \frac{1}{10-x}dx$$
$$= \lim_{t\to 10^-} -\ln(10-x)\Big|_0^t$$
$$= \lim_{t\to 10^-}(\ln(10) - \ln(10-t)) = \infty.$$

(c)

$$\int_0^1 \ln(x)\,dx = \lim_{t\to 0^+}\int_t^1 \ln(x)\,dx$$
$$= \lim_{t\to 0^+}(x\ln(x) - x)\Big|_t^1$$
$$= \lim_{t\to 0^+}[(1\ln(1) - 1) - (t\ln(t) - t)]$$
$$= -1$$

since

$$\lim_{t\to 0^+} t\ln(t) = \lim_{t\to 0^+}\frac{\ln(t)}{t^{-1}}$$
$$(L'H)$$
$$= \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}}$$
$$= \lim_{t\to 0^+}(-t) = 0.$$

Silly 10 Point Bonus: The sequence defined recursively in Problem 6 above really does converge. Prove it does by establishing that the sequence is bounded, with an explicit appropriate bound, and monotone, in a suitable sense. Yes an induction argument or two is in the offing. Tell me where your work is, for there isn't room here. Solution in "gr-t3-b.pdf."