
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

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6. (5 pts.) Assume that the sequence $\{x_n\}$ is defined recursively by the formula

$$x_{n+1} = (5 + x_n)^{1/2} \text{ if } n \geq 1.$$

If $x_1 = 1$ and if $L = \lim_{n\to\infty} x_n$ exists, what is the exact value of L?? Since

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (5 + x_n)^{1/2} = (5 + \lim_{n\to\infty} x_n)^{1/2} = (5 + L)^{1/2},$$

the limit L is a solution to the quadratic equation

$$L^2 - L - 5 = 0.$$

Thus,using the quadratic formula, it follows that

$$L = \frac{1 + (21)^{1/2}}{2} \text{ or } L = \frac{1 - (21)^{1/2}}{2}.$$

Since the sequence is non-negative, the second root is not a possible value for the limit.

Silly 10 Point Bonus: The sequence defined recursively in Problem 6 above really does converge. Prove it does by establishing that the sequence is bounded, with an explicit appropriate bound, and monotone, in a suitable sense. Yes an induction argument or two is in the offing. Tell me where your work is, for there isn't room here.

Okay, I fibbed. We shall actually do three simple induction arguments to show

(a) $x_n \geq 1$ for each integer $n \geq 1$;
(b) $x_n < x_{n+1}$ for each integer $n \geq 1$; and
(c) $x_n < 3$ for each integer $n \geq 1$.

Plainly, (b) and (c) imply that the sequence $\{x_n\}$ is increasing and bounded above, and thus, convergent. (a) is thrown in just to complete the solution of Problem 6. That the sequence really is non-negative really should be proved.

(a): Basis Step: Since $x_1 = 1 \geq 1$ from the definition of the sequence in Problem 6, the basis step of the induction is true.

Induction Step: We must show that if n is an arbitrary positive integer, then $x_n \geq 1$ implies that $x_{n+1} \geq 1$. Thus, pretend that $x_n \geq 1$. It then follows from the recursive definition of the sequence, the induction hypothesis, and the observation that the square root function is increasing that

$$x_{n+1} = (5+x_n)^{1/2} \geq (5+1)^{1/2} \geq 1.$$

Thus, we have shown that if n is an arbitrary positive integer, then $x_n \geq 1$ implies that $x_{n+1} \geq 1$. It follows that

$$x_n \geq 1 \text{ implies that } x_{n+1} \geq 1$$

for every positive integer n. This completes the proof of the induction step.

To complete the proof of (a), we need only appeal to the Principle of Math Induction.//

(b): Basis Step: From the definition of the sequence, $x_1 = 1$. The recursive definition provides us with

$$x_1 = 1 < (6)^{1/2} = (5+x_1)^{1/2} = x_2.$$

Induction Step: We must show that if n is an arbitrary positive integer, then $x_n < x_{n+1}$ implies that $x_{n+1} < x_{(n+1)+1}$. Thus, pretend n is an arbitrary positive integer and that we have $x_n < x_{n+1}$. Then, using the recursive definition of the sequence,

$$\begin{aligned} x_n < x_{n+1} &\Rightarrow \quad 5+x_n < 5+x_{n+1} \\ &\Rightarrow \quad (5+x_n)^{1/2} < (5+x_{n+1})^{1/2} \\ &\Rightarrow \quad x_{n+1} < x_{(n+1)+1}. \end{aligned}$$

Thus, we have shown that if n is an arbitrary positive integer, then $x_n < x_{n+1}$ implies that $x_{n+1} < x_{(n+1)+1}$. It follows that

$$x_n < x_{n+1} \text{ implies that } x_{n+1} < x_{(n+1)+1}$$

for every positive integer n. This completes the proof of the induction step.

To complete the proof of (b), we need only appeal to the Principle of Math Induction.//

(c): Basis Step: Since $x_1 = 1 < 3$ from the definition of the sequence in Problem 6, the basis step of the induction is true.

Induction Step: We must show that if n is an arbitrary positive integer, then $x_n < 3$ implies that $x_{n+1} < 3$. Thus, pretend n is an arbitrary positive integer and that we have $x_n < 3$. Then, using the recursive definition of the sequence,

$$\begin{aligned} x_n < 3 &\Rightarrow \quad 5+x_n < 5+3 < 9 \\ &\Rightarrow \quad (5+x_n)^{1/2} < (9)^{1/2} \\ &\Rightarrow \quad x_{n+1} < 3. \end{aligned}$$

Thus, we have shown that if n is an arbitrary positive integer, then $x_n < 3$ implies that $x_{n+1} < 3$. It follows that

$$x_n < 3 \text{ implies that } x_{n+1} < 3$$

for every positive integer n. This completes the proof of the induction step.

To complete the proof of (c), we need only appeal to the Principle of Math Induction.//

Question: Why did I use "3" as an upper bound instead of the number "$(1 + (21)^{1/2})/2$" which is the least upper bound?? Try proving (c) with 3 replaced by $(1 + (21)^{1/2})/2$. 'Tis a little messier.