
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

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1. (10 pts.) Find Taylor's formula for the given function f at a = π. Find both the Taylor polynomial, $P_3(x)$, and the Lagrange form of the remainder term, $R_3(x)$, for the function f(x) = sin(x) at a = π. Then write ~~cos(x)~~ sin(x) in terms of $P_3(x)$ and $R_3(x)$.

$$P_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(\pi)}{k!}(x-\pi)^k = \cos(\pi)(x-\pi) - \frac{\cos(\pi)}{3!}(x-\pi)^3$$
$$= -(x-\pi) + \frac{1}{6}(x-\pi)^3.$$

$$R_3(x) = \frac{f^{(4)}(c)}{4!}(x-\pi)^4 = \frac{\sin(c)}{24}(x-\pi)^4$$

for some c between x and π.

$$\sin(x) = -(x-\pi) + \frac{1}{6}(x-\pi)^3 + \frac{\sin(c)}{24}(x-\pi)^4$$

for some c between x and π.

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2. (10 pts.) (a) Find the rational number represented by the following repeating decimal.

0.2121212121 ... = 21/99 = 7/33.

This may be obtained by the "High School" method or by summing

the infinite series $\sum_{k=1}^{\infty} \frac{21}{(10^2)^k} = \ldots = \frac{21}{100}\left[\frac{1}{1-\left(\frac{1}{100}\right)}\right] = \ldots .$

(b) Find all values of x for which the given geometric series converges, and then express the closed form sum of the series as a function of x.

$$\sum_{k=1}^{\infty} \frac{(x+1)^k}{5^k} = \sum_{j=0}^{\infty} \frac{(x+1)^{j+1}}{5^{j+1}} = \sum_{j=0}^{\infty} \frac{(x+1)}{5}\cdot\left(\frac{x+1}{5}\right)^j = \frac{x+1}{4-x}$$

provided $|(x + 1)/5| < 1$ or $|x + 1| < 5$. As an interval what you have in hand is I = (-6, 4).

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3. (10 pts.) Using either comparison test or limit comparison test, determine whether each of the following series converges.

(a) $\sum_{n=1}^{\infty} \frac{3n^2+5}{4n + n^5}$ Since

$$\frac{3n^2+5}{4n + n^5} \le \frac{3}{n^3}$$

for $n \ge 1$, and $\sum_{n=1}^{\infty} \frac{3}{n^3}$ is a convergent p-series, comparison test implies that the series in (a) converges.

(b) $\sum_{n=1}^{\infty} \frac{24n-1}{n^2+n}$ Since

$$\frac{1}{n} \le \frac{24n-1}{n^2+n}$$

for $n \ge 1$, and $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, is divergent, comparison test implies that the series of (b) diverges.

You may also use the limit comparison test here to deal with either or both of (a) and (b).

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4. (10 pts.) (a) Explain very briefly why integral test may not be used to show the series below is convergent.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

First, the series is neither a positive term nor a negative term series, nor are its terms eventually positive or negative. We can see this easily since the family of open intervals $I_k = ((2k-1)\pi, 2k\pi)$ and $J_k = (2k\pi, (2k+1)\pi)$ for $k \ge 1$ are all of length π, and thus must contain positive integers. Plainly $\sin(x) < 0$ for $x \in I_k$ and $\sin(x) > 0$ for $x \in J_k$. Second, with a bit more work, you can show that $f(x) = \sin(x)/x^2$ is not decreasing from some point onward.

(b) Using only the integral test, determine whether the series below converges. Be explicit about the definition of the function f(x) used, and verify all the hypotheses of the theorem are true. [Warning: Details, details, details ...]

$\sum_{n=1}^{\infty} \frac{2n}{n^4+1}$ Let $f(x) = 2x/(x^4 + 1)$ for $x \ge 1$. Then f is a positive, continuous function. Since $f'(x) = (2 - 6x^4)/(x^4+1)^2$, $f'(x) < 0$ for $x > 1$. Thus, f is decreasing for $x \ge 1$. Since

$$\int_1^{\infty} \frac{2x}{x^4+1}dx = \lim_{b\to\infty} \int_1^{b} \frac{2x}{x^4+1}dx = \lim_{b\to\infty}[\tan^{-1}(b^2)-\tan^{-1}(1)] = \frac{\pi}{4}$$, integral test implies that the given series converges.

5. (10 pts.) Find the sums of each of the following convergent series. [Pay attention to the lower limits of summation, Folks.]

(a) $$\sum_{n=2}^{\infty} \frac{1}{10^n} = \sum_{n=2}^{\infty} \left(\frac{1}{10}\right)^n = \sum_{j=0}^{\infty} \left(\frac{1}{10}\right)^{(j+2)} = \ldots = \left(\frac{1}{100}\right)\left(\frac{1}{1 - \left(\frac{1}{10}\right)}\right) = \frac{1}{90}.$$

(b) $$\sum_{n=1}^{\infty} \frac{12}{n(n+2)} = \lim_{N\to\infty} 6\left(1 - \frac{1}{N+1} + \frac{1}{2} - \frac{1}{N+2}\right) = 9 \text{ since}$$

$$\sum_{n=1}^{N} \frac{12}{n(n+2)} = \sum_{n=1}^{N}\left(\frac{6}{n} - \frac{6}{n+2}\right) = 6\sum_{n=1}^{N}\left[\left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right]$$

$$= 6\left[\left(1 - \frac{1}{N+1}\right) + \left(\frac{1}{2} - \frac{1}{N+2}\right)\right],$$

once you recognize the telescoping tormenters.

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6. (5 pts.) Using divergence test, show that the series

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+1}\right)$$

diverges. [Hint: A simple complex sentence with an embedded computation produces the desired magic.]

Since $\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = 1$ implies that $\lim_{n\to\infty}(-1)^n\left(\frac{n}{n+1}\right)$ doesn't exist,

divergence test implies that $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+1}\right)$ diverges.

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7. (5 pts.) It turns out the infinite series

$$\sum_{n=1}^{\infty} \frac{15}{n^4}$$

converges and has a sum that we shall denote by S. If you want a numerical estimate of S that is accurate to 2 decimal places, which partial sum,

$$S_N = \sum_{n=1}^{N} \frac{15}{n^4},$$

can you prove does the job? [Hint: There is an improper integral that provides an upper bound on the true error.]
Since

$$|\, S - S_N \,| < \int_N^{\infty} \frac{15}{x^4}\, dx = \frac{5}{N^3}$$

for $N \geq 1$ by using the error estimate from integral test, to obtain 2 decimal place accuracy, it suffices to have $5/N^3 \leq (1/2)10^{-2}$. This last inequality is equivalent to $10 \leq N$. Thus, the sum with N = 10 will do the job.

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8. (10 pts.) Find the radius of convergence and the interval of convergence of the power series function

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x-3)^k}{(k+1)10^k} .$$

First observe that the series is centered at x = 3. Then we apply the ratio test for absolute convergence in order to determine the radius of convergence of the power series.

$$\rho(x) = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \ldots = \lim_{k\to\infty} \frac{1}{10}\frac{k+1}{k+2}|x-3| = \frac{1}{10}|x-3|.$$

Now, $\rho(x) < 1$ if, and only if $|x - 3| < 10$. Thus, R = 10 is the radius of convergence. The endpoints are $x_L = -10 + 3$ and $x_R = 3 + 10$. When you substitute x_L into the power series and simplify the algebra, you obtain $\sum_{k=0}^{\infty} \frac{1}{k+1}$ which diverges. When you substitute x_R into the power series and simplify the algebra, you obtain $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ which converges. I = (-7, 13].

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9. (10 pts.) (a) Apply the alternating series remainder estimate to estimate the error in approximating the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$$

by the sum

$$\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n^3} .$$

[Simply write an appropriate inequality.]

If we denote the sum of the series by S and the partial sum above by S_5, then the error estimate from the alternating series test implies that $|S - S_5| < 1/(6)^3 = 1/216$. Of course you may also write this as $|R_5| < 1/216$.

(b) Find a positive integer N such that the partial sum

$$\sum_{n=1}^{N} (-1)^{n+1}\left(\frac{5}{n^5}\right)$$

approximates the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{5}{n^5}\right)$$

to 4 decimal places, and prove your N actually does what you claim.

Let S denote the sum of the series and S_N denote the Nth partial sum above. Then the error estimate from the alternating series test implies that $|S - S_N| < 5/(N+1)^5$. Consequently, to obtain 4 decimal place accuracy, it suffices to find a positive integer N so that $5/(N+1)^5 \le (1/2)10^{-4}$. Now this last inequality is equivalent to $10^5 \le (N+1)^5$, or $9 \le N$. N = 9 does the job.//

10. (15 pts.) (a) Using known power series, obtain a power series representation for the function f(x) = (x - arctan(x))/x^3. Write your answer using sigma notation.

$$f(x) = \frac{x - \arctan(x)}{x^3} = \frac{1}{x^3}\left[x - \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}\right]$$

$$= \frac{1}{x^3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k+1}x^{2k+1} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k+1}x^{2k-2} = \sum_{j=0}^{\infty}\frac{(-1)^j}{2j+3}x^{2j}.$$

(b) Find a power series representation for the function f(x) below by doing termwise integration. Write your answer using sigma notation.

$$f(x) = \frac{\pi}{2} - \int_0^x \frac{1}{1+t^2}\,dt = \frac{\pi}{2} - \int_0^x \sum_{k=0}^{\infty}(-1)^k t^{2k}\,dt$$

$$= \frac{\pi}{2} - \sum_{k=0}^{\infty}(-1)^k\int_0^x t^{2k}\,dt = \frac{\pi}{2} - \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1}$$

for -1 < x < 1.

(c) Beginning with the power series function

$$f(x) = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k x^k$$

defined for x satisfying $|x| < 2$, differentiate termwise to find the series representation for f′(x). Write your answer using sigma notation.

$$f'(x) = \sum_{k=0}^{\infty}\frac{d}{dx}\left[\left(\frac{1}{2}\right)^k x^k\right] = \sum_{k=1}^{\infty}k\left(\frac{1}{2}\right)^k x^{k-1}.$$

also for x satisfying $|x| < 2$. Why "k=1"?? Remember our gentle person’s agreement with respect to constant terms and sigma notation???

11. (5 pts.) With proof, determine whether the given series is conditionally convergent, absolutely convergent, or divergent.

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}$$

Since the series of absolute values is given by

$$\sum_{k=1}^{\infty}\frac{1}{k^2},$$

a convergent p-series, the original series is absolutely convergent

10 Point Bonus: (a) The power series function f(x) in 10(b) above may be written in terms of an old friend. Do so and reveal the *true identity* of f(x) in 10(b). (b) Obtain an infinite series that converges to the exact value of arctan(2). [Say where your work is. It won’t fit here.] c2-t4-b.pdf has this.