Student Number: **Exam Number:**

Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all the magic on the page. Eschew obfuscation.*

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1. (72 pts.) Obtain the exact numerical value of each of the following if possible. If, by chance, a limit fails to exist or an improper integral or an infinite series fails to converge, say so. Watch your step. Some definite integrals have teeth.
9 points/part.

(a) $\int_0^{(\pi/4)^{1/2}} 4\theta\cos(\theta^2)\, d\theta =$

(b) $\int_0^{\pi/4} 4\theta\sin(\theta)\, d\theta =$

(c) $\int_{10}^{\infty} \frac{20}{x^2+x}\, dx =$

(d) $\sum_{k=10}^{\infty} \frac{20}{k^2+k} =$

1. Obtain the exact numerical value of each of the following if possible...

(e) $\lim_{n \to \infty} \left(1 + \frac{\pi}{n}\right)^n =$

(f) $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sec^2 \left(\frac{4}{\pi} \left(\frac{k}{n} \right) \right) =$

(g) $\sum_{k=1}^{\infty} 25 \left(\frac{4}{5} \right)^k$

(h) $\int_0^1 \frac{2x}{(1-x^2)^{1/2}} \, dx =$

2. (36 pts.) Here are six easy antiderivatives to evaluate.

(a) $\displaystyle\int \frac{\sin^2(x)}{\cos(x)}\,dx =$

(b) $\displaystyle\int \ln(x+1)\,dx =$

(c) $\displaystyle\int \sin^2(x)\,dx =$

(d) $\displaystyle\int \frac{4}{x^2-1}\,dx =$

(e) $\displaystyle\int 2xe^{2x}\,dx =$

(f) $\displaystyle\int \frac{4x^4+4x^2+36x+1}{x^2+1}\,dx =$

3. (18 pts.) Suppose that

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 7^n}(x - 1)^n$$

(a) Find the radius of convergence and the interval of convergence of the power series function f.

(b) By using sigma notation and doing termwise differentiation, obtain a power series for $f'(x)$. What is the radius of convergence of the series for $f'(x)$?

$f'(x) =$

(c) Obtain an infinite series whose sum is the same as the numerical value of the following definite integral. [We are working with the function f of part (a) of this problem. Integrate termwise. **Use sigma notation.**]

$$\int_1^2 f(x)dx =$$

4. (18 pts.) (a) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc. Do not attempt to integrate the rational function.

$$\frac{x^2+81}{4x(x+1)^2(x^2+1)^2} =$$

(b) If one were to integrate the rational function in part (a), at some point one would have to deal with the indefinite integral below. Reveal, in detail, how to evaluate this integral.

$$\int \frac{1}{(x^2+1)^2}\,dx =$$

5. (12 pts.) Each of the following power series functions is the Maclaurin series of some well-known function. In each case, (i) identify the function, and (ii) provide the interval in which the series actually converges to the function.

(a) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} =$

(b) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} =$

(c) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} =$

(d) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} =$

(e) $\sum_{k=0}^{\infty} \frac{x^k}{k!} =$

(f) $\sum_{k=0}^{\infty} x^k =$

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6. (24 pts.)
(a) (18 pts.) Find both the 2^{nd} Taylor polynomial at $a = 8$, $P_2(x)$, and the Lagrange form of the remainder term, $R_2(x)$, for the function $f(x) = (1 + x)^{1/2}$. Then write $(1 + x)^{1/2}$ in terms of $P_2(x)$ and $R_2(x)$.

$P_2(x) =$

$R_2(x) =$

$(1 + x)^{1/2} =$

(b) (6 pts.) Using part (a), provide a reasonable estimate of how many decimal places accuracy your computation would provide if you used $P_2(x)$ to approximate $(1 + x)^{1/2}$ with $|x - 8| \le 0.1$.

7. (20 pts.) (a) State the Fundamental Theorem of Calculus.

(b) Compute $g'(x)$ when $g(x)$ is defined by the following equation.

$$g(x) = \int_0^x \sin^3(t)\,dt + x^3$$

$g'(x) =$

(c) Write the solution to the following initial value problem in terms of a definite integral with respect to the variable t, but don't attempt to evaluate the definite integral:

$$y'(x) = e^{\sin(\pi x)}, \quad y(0) = \pi/3.$$

$y(x) =$

(d) Suppose that $f(0) = \pi$ and

$$f'(x) = \frac{8x^2}{(1 - x^2)^{1/2}} .$$

What function f(x) satisfies these two equations?? [Identify f as completely as possible. f can be written in terms of an old friend.]

$f(x) =$

Silly 20 point bonus: Show how to obtain a rational number that approximates $\tan^{-1}(2)$ to 6 decimal places, with proof the process works. Where's your work??